Turbulence and Spatial Correlation of Currents in Quantum Chaos

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We have found spatial decoherence (loss of phase coherence between distant points in space) of quantum states and chaotic behavior (“turbulence”) of quantum-mechanical currents for a nonlinear oscillator in a magnetic field. Use of quantum currents is dictated by the fact that they are simultaneously phase-sensitive and gauge-invariant measures. The theory is applicable to an electron in a quantum dot with soft confining potential in a magnetic field. [S0031-9007(98)07722-9]

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Fundamental properties of quantum chaos [1] became experimentally observable in a variety of physical systems. One of the most actively studied classes of such systems are semiconductor heterostructures [2,3]. Quantum chaos affects in a dramatic way transport properties of quantum wells and dots. Microscopically, transport and other responses depend on corresponding correlation functions [3,4]. Loss of correlation of quantum-mechanical amplitude in temporal domain (quantum decoherence) is one of the signatures of quantum chaos that is important for transport properties [5].

In this paper we find spatial quantum decoherence, i.e., loss of spatial correlation of the phase of a chaotic wave function, for an electron in a nonlinear oscillator potential in a magnetic field. A significant problem that a magnetic field causes is due to the fact that the quantum phase is not a gauge-invariant measure. We overcome this by considering correlation functions of quantum-mechanical currents that are both gauge invariant and phase sensitive. We show that currents in individual eigenstates and their averaged spatial-correlation functions are sensitive measures of quantum chaos. In the region of developed chaos the currents become chaotic (“turbulent”), and the correlation between currents in two distant spatial point decays is reduced. The system considered is a model of quantum dots with a soft potential in a magnetic field. The correlation functions found are related to magnetic and optical responses of the system [4].

Spatial-correlation properties of quantum-chaotic states have been a subject of intense interest for over two decades [1]. A pioneering paper by Berry [6] suggested that spatial correlation of wave functions of chaotic states decays by a power law. This has been confirmed numerically [7]. One can obtain indirect information about quantum phases studying the correlation of probability densities as in Refs. [8,9]. It has been shown in the supersymmetry approach that magnetic fields can induce long-range spatial correlations for quantum-chaotic states [9]. We do not find evidence supporting this.

One can specify the information that is obtainable from the current $j$ in the case of a weak (electro)magnetic field, where the perturbation theory yields the well-known Kubo-type formula (see, e.g., [4])

$$
\langle j_a(x) \rangle = \int \Pi_{a\beta}(x,x') A_{\beta}(x') dx',
$$

$$
\Pi_{a\beta}(x,x') = \frac{i}{\hbar c} \theta(t - t') \times (\langle j_a(r,t), j_{\beta}(r',t') \rangle) A_{\beta}(r',t'),
$$

where $\Pi_{a\beta}(x,x')$ is the retarded Green’s function of currents, $x = (r,t), x' = (r',t')$. $A_{\beta}(r',t')$ is a vector potential of the external field, and all operators are in the interaction representation. Here and throughout the paper, Greek subscripts denote vector indices, with summation over repeated indices implied.

In our case of a stationary field, the expression for current, averaged over states in a narrow interval at an energy $E$, reduces to

$$
\langle j_a(r) \rangle_E = \frac{1}{c} \int C_{a\beta}(r,r';E) A_{\beta}(r') dr',
$$

$$
C_{a\beta}(r,r';E) = \sum_{m \perp n} \frac{1}{E_{nm}} \langle n | j_a(r) | m \rangle \langle m | j_{\beta}(r') | n \rangle \delta(E - E_n - \delta(E - E_m)),
$$

where $| n \rangle$ and $E_n$ are the exact eigenstates and the corresponding energies of the system, and $\delta(\ldots)$ is Dirac’s delta function. For a uniform magnetic field $B$, as in our case, this relation further simplifies,

$$
\langle j(r) \rangle_E = \frac{1}{c} B_a \sum_{m \neq n} \frac{1}{E_{nm}} \langle n | j(r) | m \rangle \langle m | L_a | n \rangle \delta(E - E_n - \delta(E - E_m)),
$$

where $L$ is the angular momentum. As it follows from Eqs. (2) and (3), the averaged current in response to the external field is determined by a correlator $C_{a\beta}(r,r';E)$ of the quantum-mechanical currents, which further reduces to a current-angular momentum correlator for a uniform magnetic field.

The current correlator $C_{a\beta}(r,r')$ in Eqs. (2) and (3) is determined only by transition currents, i.e., by nondiagonal matrix elements of $j$ that exist even without a magnetic
field. However, if a magnetic field is present, then mean (diagonal) currents may exist. Their correlator, though not related to any linear response, is a direct gauge-invariant measure of the phase coherence in the corresponding eigenstates. Accordingly, we define a function $S_{ab}(r, E)$ that gives correlation of currents at two spatial points separated by a distance of $r$ averaged over states at an energy $E$,

$$
S_{ab}(r, E) = \frac{1}{\rho(E)} \sum_{n} \int d\mathbf{r}_1 d\mathbf{r}_2 \langle n| j_a(\mathbf{r}_1)|n \rangle \times \langle n| j_b(\mathbf{r}_2)|n \rangle \delta(\mathbf{r} - \mathbf{r}_{12}) \delta(E - E_n),
$$

where $\rho(E) = \sum_n \delta(E - E_n)$ is the density of states, and $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$. Using a central symmetry of the system, we consider a scalar distribution $S(r, E) = \frac{2\pi}{r} r S_{ab}(r, E)$. In numerical computations, the current conservation is not a trivial identity because it crucially depends on the precision of the determination of the eigenstates. We have found it to be a useful, sensitive test to check in every case that $\frac{\partial}{\partial r} \langle j(r) \rangle = 0$. A useful relation follows from the fact that our system does not have any vector of the symmetry of current (odd with respect to coordinate and time reversals). Therefore, $\int \langle n| j_a(\mathbf{r})|n \rangle d\mathbf{r} = 0$. As a result, we arrive at an exact sum rule

$$
\int S_{ab}(r, E) d\mathbf{r} = \int_0^\infty S(r, E) dr = 0.
$$

This provides another test of the numerical precision.

As a chaotic system (model of a quantum dot with a soft confining potential), we consider an isotropic two-dimensional nonlinear oscillator that in the absence of a magnetic field possesses a well-known classical [10] and quantum [11] chaotic behavior. For such an oscillator in a magnetic field $\mathbf{B}$ normal to the system’s plane, the Hamiltonian in a symmetric gauge is

$$
H = H_0 + H',
$$

$$
H_0 = \frac{1}{2m} p^2 + \frac{m}{2} (\omega^2 + \omega_L^2) r^2 + \omega_L L_z,
$$

$$
H' = ar^2 \cos^2 \phi \sin^2 \phi,
$$

where $\omega$ is the oscillator’s own frequency, $L_z = -i\hbar \partial / \partial \phi$ is the angular momentum, $\omega_L = -eB/2mc$ is the Larmor frequency, and $a$ is a nonlinear coupling constant.

A familiar scale transformation to dimensionless operators $r' = r\sqrt{m\omega / \hbar}$ and $p' = p / \sqrt{\hbar m \omega}$ transforms the problem described by the Hamiltonian (6) to an equivalent problem with the following substitution of parameters: $\hbar \rightarrow 1$, $m \rightarrow 1$, $\omega \rightarrow 1$, $\omega_L \rightarrow 2$, $\beta = \omega_L / \omega$, $a \rightarrow \alpha = \hbar a / m^2 \omega^3$, and $E \rightarrow E / \hbar \omega$.

We diagonalized the Hamiltonian $H$ (6) using as a basis set eigenfunctions of $H_0$ with the same magnetic field constant $\beta$. The eigenfunctions of $H$ are expanded as

$$
\Psi_N(r, \phi) = \sum_{n,l} C_{n,l}^N \psi_{n,l}(r),
$$

$$
\psi_{n,l}(r) = \sqrt{\frac{n!}{\pi(l + n)!}} \xi^l \exp\left(-\frac{\xi^2}{2}\right) l^{(l)}(\xi) \exp(il\phi),
$$

where $C_{n,l}^N$ are expansion coefficients, the normalized radius is $\xi = r(1 + \beta^2)^{1/4}$, $l = |l|$. $l^{(l)}(\xi)$ are Laguerre polynomials [12], $l = 0, \pm 1, \pm 2, \ldots, n = 0, 1, 2, \ldots$. The size of the basis set has turned out to be of primary importance. We checked the sufficiency of this set by verifying numerically conservation of the current and validity of the sum rule (5). The typical size of basis set needed was 7000–9000 states. We used the current in an $N$th state in the form

$$
\langle N| j(\xi, \phi) | N \rangle = \sum_{n,l,n',l'} C_{n',l',n,l}^N \psi_{n',l',\xi}(\xi) \left[ \frac{e^r}{\xi} \left[ (l + 2n) \psi_{n,l}(\xi) - 2\sqrt{n(n + 1)} \psi_{n-1,l}(\xi) \right] \sin(l_{z'} - l_{z}) \phi + e^l \left[ \frac{l_{z'}}{\xi} - \frac{\beta}{\sqrt{1 + \beta^2}} \xi \right] \psi_{n,l}(\xi) \cos(l_{z'} - l_{z}) \phi \right].
$$

Most of the computations have been carried out for the magnetic field constant $\beta = 0.2$. For this value, the Poincaré mapping shows that the classical chaos develops for $E \geq 50$. The onset energy of the chaos can be estimated to scale as $E \sim (1 + \beta^2)/\alpha$. For comparison, we used higher magnetic fields, $\beta = 1.0$, where the chaos is expected for $E \geq 100$, outside of our energy range.

First, we present vector-field plots for currents. In Fig. 1 we show such a plot for an eigenstate with $E = 63$ that lies close to the beginning of the developed chaos region. The pattern of currents on the scale of the whole wave function (see the left panel) shows a superposition of a regular counterclockwise laminar current flowing along the periphery of the state and a random, turbulent current filling the central part of the plot. Magnification of this central region (see the right panel in Fig. 1) reveals that random vortices with small radii coexist with a regular current spanning the whole region.

In Fig. 2 we present the current flow for a state with $E = 78$ that corresponds classically to the case of developed chaos (there are no resonant islands corresponding to stable motion in classical Poincaré maps for this energy). Developed quantum chaos is evident in this figure. Most of the wave function on the full scale is occupied by a turbulent current, while a laminar flow is not readily observable. However, the magnification of a central region
FIG. 1. Quantum current for an eigenstate with $E = 63$ in the real space of coordinates $x$ and $y$. The direction of the current at every point is shown by an arrow and its magnitude is qualitatively indicated by the darkness of the arrow. The left panel represents the whole system and the right panel is a magnified area bound by the small rectangle in the left panel.

shown in the right panel indicates that there still exists a regular, laminar flow spanning the whole region, superimposed on a large number of vortices (short-range turbulent flows). Distinct from Fig. 1, here we see more vorticity with a smaller fraction of laminar flow.

Because currents, as linear response theory shows, are expressed in terms of the current-current correlator $C_{\alpha\beta}(\mathbf{r}, \mathbf{r}'; E)$, the chaos (turbulence) of currents implies that this correlation function is also chaotic. In this respect, the quantum-mechanical eigenstates considered remind chaotic eigenvectors (“plasmons”) of the dipolar eigenproblems found earlier [13].

The consideration of individual eigenstates, though a convincing characteristic of the spatial chaos, is not sufficient because it does not factor in a probability of finding an eigenstate with a given geometry. We complement this with a discussion of the current correlator defined by Eq. (4) that is a statistical measure of the spatial correlation averaged over many individual eigenstates.

The current-current correlator $S(r, E)$ for a nonlinear oscillator ($\alpha = 0.05$) for an intermediate magnetic field ($\beta = 0.2$) is shown in Fig. 3 for three different values of energy. For comparison, we show simultaneously on the same scale the correlator $S(r, E)$ for a linear oscillator.
FIG. 3. Current-current correlator $S(r, E)$ [Eq. (4)] for a nonlinear oscillator ($\alpha = 0.05$) and linear oscillator ($\alpha = 0$) in the same magnetic field ($\beta = 0.2$). There are 20 equal energy intervals and 50 spatial intervals in this histogram.

($\alpha = 0$) at the same magnetic field. First, we point out that the positive correlation at small distances changes to a negative correlation at a larger radius. Generally, the sign change(s) of the current correlator should always take place in view of the zero integral in Eq. (5). We verified that the value of this integral is numerically indistinguishable from zero (the relative error is less than $10^{-27}$), indicating sufficient numerical precision.

For a nonchaotic linear quantum oscillator, the correlation function in Fig. 3 does not decrease by amplitude for all energies, limited only by the total extension of the eigenstate. This manifests complete phase coherence of the regular (nonchaotic) eigenstates, as expected. A dramatically different situation exists for a chaotic nonlinear oscillator ($\alpha = 0.05$). For an energy $E = 25$, below the threshold of chaos, the effect of the nonlinearity is barely distinguishable and the wave function remains fully coherent. However, above the onset of the chaos ($E = 51$), we see that the correlation for the nonlinear oscillator is starting to decrease at both the small and large distances. This trend is strongly pronounced in the region of the developed chaos ($E = 92$). In the latter case the correlation function $S(r, E)$ for the chaotic oscillator is reduced by magnitude several times. This indicates a significant spatial decoherence of the current with establishment of quantum chaos. Because the current for a highly excited state is primarily determined by its phase, this decoherence is mostly dephasing in space due to chaos. Interestingly enough, the current correlator decreases in amplitude, but not in its spatial extension. This is in agreement with the pattern of current containing very short-range turbulent vortices superimposed on an ordered laminar flow (cf. Figs. 1 and 2).

To study the effect of increased magnetic field, we have considered (data not shown) the correlation function $S(r, E)$ for $\beta = 1$. In this case, as noted above, the system is classically nonchaotic in our range of energies. This study shows that there is very little, if any, decrease of the spatial correlation with the introduction of the nonlinear coupling in contrast to Fig. 3. This is consistent with the absence of the chaotic classical motion.

In conclusion, we have found spatial decoherence in chaotic quantum states for a nonlinear oscillator in a magnetic field. We have shown that the currents of chaotic states become highly chaotic, turbulent in the region of developed quantum chaos, reflecting this decoherence. This complements such known signs of quantum chaos as scarring of eigenfunctions [2], spectral [1], and spectral-spatial parametric [14] correlations.

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