

AC CIRCUIT ANALYSIS

Relation to DC Circuit Analysis:

In the DC circuits we considered, the currents were determined by the combination of voltage sources and resistors. Because of the linear relation between voltage and current for resistors (Ohm's law: $V=IR$), analyses using Kirchoff's laws led to systems of linear equations which were solvable by standard techniques. In addition to a straightforward application of Kirchoff's laws via loop and junction techniques, we examined three methods (superposition, nodal analysis, and Thevenin's theorem) for setting up the linear equations in a way to minimize the algebraic manipulations.

AC sources require one more descriptive term than DC sources. Specifically, in DC circuits, it is sufficient to describe the current or voltage in terms of the "amount" or amplitude. By definition, AC sources are not steady and thus the "amount" is continually changing. However, the maximum "amount," or amplitude, combined with the repetition rate, or frequency, form a suitable combination of descriptive terms. (It may also be useful to specify the "waveform": sinusoidal, square, triangular, sawtooth, etc. Our focus will be on the sinusoidal shape.) Thus, AC introduces frequency as the additional descriptive term.

If the only circuit elements are resistors and voltage sources, then AC circuits can be analyzed with exactly the same techniques as DC circuits. However, capacitors and inductors are also important circuit elements, and we need to introduce them at this point. To figure out what modifications may be needed to our circuit analysis methods when capacitors and / or inductors are present, we need to recall the relations between voltage and current:

$$\text{Resistor: } V_R = I_R R$$

$$\text{Capacitor: } V_C = Q_C / C = \left(\int I_C dt \right) / C$$

$$\text{Inductor: } V_L = -L(dI_L / dt)$$

Obviously, the voltage-current relations for capacitors and inductors are not linear. Nevertheless, for sinusoidal waveforms, there is kind of linearity between these parameters. Using capacitors as the example, when $I_C = I_{C0} \sin \omega t$, $V_C = I_{C0}(-1/\omega t) \cos \omega t = V_{C0} \cos \omega t$. Thus, $V_{C0} = I_{C0}(-1/\omega t)$ and the linear relation is between the **amplitudes** of the voltage and current. This result is the basis of **reactance**, a quantity which is used for both capacitance and inductance; *i.e.*, **capacitive reactance** = $X_C = (V_{C0} / I_{C0}) = (1/\omega C)$.

Although this helps with the problem of using the DC-based procedures with AC circuit

analysis, it does not completely solve the problem. Specifically, the amplitude of an AC waveform is its maximum or peak, and the peak of the voltage waveform (a sine function in the example above) occurs at a **different time** than the peak of the current waveform (a cosine function in the example). In other words, for both capacitors and inductors, the voltage and current are **90° (or $\pi/4$) out of phase**.

Because the current and voltage may be out of phase in AC circuits, the analytical methods must be able to keep track of phase as well as amplitude. (In the circuits, the frequency property of AC develops into the phase property; *i.e.*, the necessity to keep track of phase is the consequence of frequency.) Thus, the algebra / arithmetic used for AC circuit analysis must have the capability of dealing with **two parameters at the same time**. Mathematics has several tools for doing this. For example, **vectors** have both magnitude and direction, so vector algebra / arithmetic has this capability. In addition, **complex numbers** have a real and an imaginary part, so complex algebra / arithmetic also has the suitable character.

In this course, and in the larger field of electrical engineering, the method most often chosen is based on complex numbers. A basic reason for doing so is that division is a commonly-needed operation in AC circuit analysis; while division is a well-defined operation for complex algebra, it is not a defined operation for vectors.

In summary, therefore, the concept of resistance ($R=V/I$) in DC circuits is extended to the concept of **impedance** in AC analysis. Impedance is a complex number in general and thus carries both amplitude and phase information. For resistors, the impedance is the resistance; for capacitors and inductors, the impedance is built from the reactance (X_L and X_C). A summary of impedance descriptions for the three basic circuit elements is listed below in the table (N.B. $\omega = 2\pi f$):

Element	Impedance
Resistor	$Z_R = R$
Capacitor	$Z_C = -jX_C = -j(1/\omega C)$
Inductor	$Z_L = jX_L = j(\omega L)$

Moreover, impedance “behaves” like resistance. That is, the series and parallel combination rules are the same, AC “Ohm’s law” is $V = IZ$, etc. With these properties and definitions, the same methods used for DC circuit analysis can be applied to AC analysis: loop analysis, superposition, nodal analysis, and Thevenin’s theorem. All that is necessary for completing the analysis and obtaining an “answer” is suitable application of complex number algebra and arithmetic.

Example 1:

$$Z_{eq} = Z_1 + Z_2 + Z_3$$

Sample Numbers:

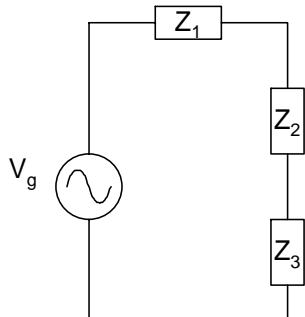
$$Z_1 = 500\Omega = 0.5k$$

$$I_g = V_g / Z_{eq}$$

$$Z_2 = -j750\Omega = -j0.75k$$

$$Z_3 = j500\Omega = j0.5k$$

$$Z_{eq} = 0.5k - j0.75k + j0.5k = 0.25k(2 - j)$$



$$I_g = \frac{20V}{0.25k(2 - j)} = \frac{80}{5}(2 + j) \text{ ma} = 16(2 + j) \text{ ma}$$

Also: $V_1 = I_g Z_1 = [16(2 + j)\text{ma}](0.5k) = 8(2 + j) \text{ V}$

$$V_2 = I_g Z_2 = (-j0.75)(16)(2 + j)\text{V} = 12(1 - j2)\text{V}$$

$$V_3 = I_g Z_3 = (j0.5k)(16)(2 + j) \text{ V} = 8(-1 + j2)\text{V}$$

Note that:

$$V_1 + V_2 + V_3 = [(16 + 12 - 8) + j(8 - 24 + 16)]\text{V} = 20\text{V} = V_g$$

Note also that:

$$|V_1| = 8\sqrt{5}\text{V} = 17.89\text{V}$$

$$|V_2| = 12\sqrt{5}\text{V} = 26.83\text{V}$$

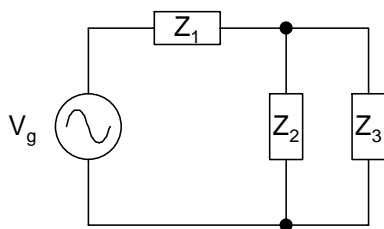
$$|V_3| = 8\sqrt{5}\text{V} = 17.89\text{V}$$

Obviously, $|V_1| + |V_2| + |V_3| \neq |V_g|$; in fact, $|V_2| > |V_g|$ all by itself for this example!!!

Example 2:

$$Z_{eq} = Z_1 \text{ (series) } (Z_2 \parallel Z_3)$$

$$= Z_1 + \frac{Z_2 Z_3}{Z_2 + Z_3}$$



$$I_g = V_g / Z_{eq}$$

For the values:

$$V_g = 20\text{V}, Z_1 = 0.5k, Z_2 = -j0.75k, Z_3 = j0.5k,$$

This becomes:

$$Z_{eq} = 0.5k + \frac{(j0.5k)(-j0.75k)}{j0.5k - j0.75k} = 0.5k + j1.5k = 0.5k(1 + j3)$$

$$I_g = \frac{20V}{0.5k(1+j3)} = \frac{40}{10}(1-j3)ma = 4(1-j3)ma;$$

$$V_1 = I_g Z_1 = 0.5k(4)(1-j3)ma = 2(1-j3)V$$

$$I_2 = \frac{V_g - V_1}{Z_2} = \frac{20 - 2(1-j3)V}{-j0.75k} = \frac{(18+j6)V}{-j0.75k} = 8(-1+j3)ma$$

$$I_3 = \frac{V_g - V_1}{Z_3}, \quad \text{or}$$

$$I_3 = I_g - I_2, \quad \text{etc.}$$

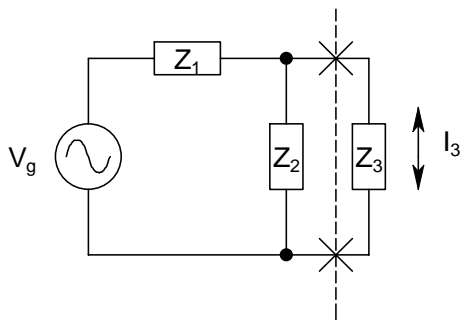
Use of Thevenin's Theorem in AC Circuit Analysis:

Thevenin's theorem is particularly important in general since it emphasizes the existence of simplified equivalent circuits. Equivalent circuits are central to the approach of using model circuits to describe and predict the behavior of electric and electronic circuits.

The following example, simply the circuit of **example 2** with a different set of component values, illustrates Thevenin's theorem applied to AC circuit analysis as we did with DC circuits.

Example 3:

$$V_g = 20V, \quad Z_1 = -j0.75k, \quad Z_2 = j0.5k, \quad Z_3 = 0.5k$$



Question: $I_3 = ?$

Dividing the circuit as shown for use of Thevenin's theorem yields the following for Z_{th} and V_{th} :

$$\begin{aligned} Z_{th} &= Z_1 \parallel Z_2 = Z_1 Z_2 / (Z_1 + Z_2) \\ &= (-j0.75k)(j0.5k) / (-j0.75k + j0.5k) \\ &= 3(j0.5k) = \mathbf{j1.5k} \end{aligned}$$

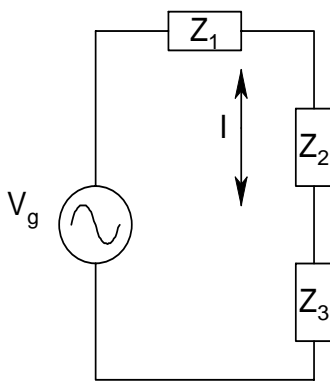
$$V_{th} = I' Z_2 = \left(\frac{V_g}{Z_1 + Z_2} \right) Z_2 = \left(\frac{20}{-j0.25k} \right) (j0.5k) = \mathbf{-40V}$$

$$\therefore I_3 = \frac{V_{th}}{Z_{th} + Z_3} = \frac{-40V}{0.5k(1+j3)} = \mathbf{8(j3-1)ma}$$

Algebraic Voltage Differences with AC:

With DC circuits, it was important to set up the analysis by picking a current direction to establish the “high-voltage” side of resistors. While the physics behind this is also important with AC circuits, the periodic reversal of current makes the concept of **direction** ambiguous. (Actually, the relative signs of algebraic voltage differences in AC are expressed as the relative phases.)

The AC procedures we are setting up take care of the bookkeeping automatically as follows:



1. When represented by impedances, voltage differences are always **voltage drops** (expressed as negative) for setting up loop equations:

$$\text{Loop Equation: } 0 = V_g - IZ_1 - IZ_2 - IZ_3$$

2. Impedance **descriptions** for R, L, and C incorporate “+’s” and “-’s” along with the “j’s” to take care of relative phases. (Note the “-” in $Z_C = -j/\omega C$.)

Interpretation of Impedance in the $|Z|$, ϕ Representation:

Recall that the purpose of complex impedance is to manage the two parameters needed in AC analysis, the ratio of voltage and current amplitudes and their phase relation. The meaning of $|Z|$ is straightforward: it expresses the ratio of voltage and current amplitudes, $|Z| = V_0 / I_0$. The angle ϕ is most easily interpreted using the exponential form of complex numbers:

$$Z = \frac{V}{I} = \frac{|V|e^{j\phi_V}}{|I|e^{j\phi_I}} = \frac{V_0}{I_0} e^{j(\phi_V - \phi_I)} = |Z| @ \phi = (\phi_V - \phi_I)$$

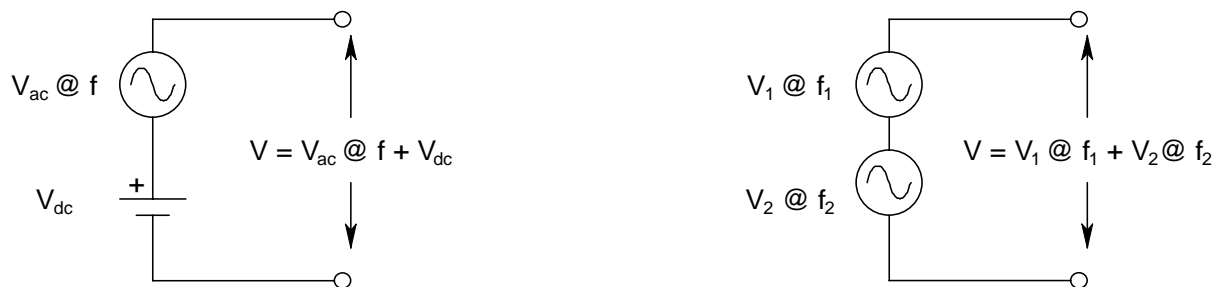
Thus, the “angle” described by complex impedance expresses the phase relative angle between the voltage and current for the impedance. Specifically, if $\phi > 0$, the voltage is ahead of (**leads**) the current; if $\phi < 0$, the voltage is behind (**lags**) the current.

Multiple Voltage Sources:

Although AC circuit analysis with multiple sources is straightforward, it is necessary to consider two points unnecessary in DC circuits. First, the sources may operate at different frequencies. The second is that sources even at the same frequency may be out of phase. For DC, all sources are at the same frequency ($f = 0$) and phase, and appropriate combination of DC sources into one equivalent unit is simple addition. For AC sources at the same frequency but different phase, the combination requires use of complex (or

phasor) analysis for expressing the voltages in the rectangular form before addition. For AC sources at different frequencies, about the best “single unit” combination is expression of the combined voltages with terminology such as “ $V = V_1 @ f_1 + V_2 @ f_2$.” Note that this is the general idea of superposition.

We will not study further AC circuits with multiple sources. However, it will be useful to consider the case of voltages having **components** at different frequencies. Two examples illustrated with circuits are:



Power in AC circuits:

In our discussion of DC we derived the basic relation for electric power:

$$P = VI.$$

Since V and I in general are continuously changing with AC, we must face the question of what V and I values are appropriate for calculating the power. From this situation comes the concept of **average power**. As the values are periodically repetitive average power needs only to be calculated from one cycle as follows:

$$P_{av} = \frac{1}{T} \int_0^T V(t)I(t)dt, \quad T = \text{Period of waveform}$$

The exact form of $V(t)$ and $I(t)$ depend on the waveform, and thus the result of the integration is waveform-dependent. For our case, that of sinusoidal waveforms, the result is as follows:

$$I(t) = I_0 \sin \omega t; \text{ As a result of } V = IZ,$$

$$V(t) = V_0 \sin(\omega t + \phi).$$

$$P = \frac{V_0 I_0}{T} \int_0^T \sin(\omega t + \phi) \sin \omega t dt$$

For a resistor, $\phi = 0$, so $\sin(\omega t + \phi) = \sin \omega t$; for both capacitors and inductors, $\phi = 90^\circ$,

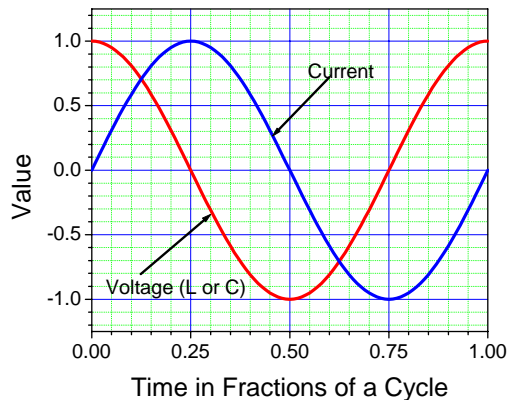
so $\sin(\omega t + \phi) = \cos\omega t$. The results are as follows:

For resistors, recalling that $\omega = 2\pi f = \frac{2\pi}{T}$:

$$P = \frac{V_0 I_0}{T} \int_0^T \sin^2 \omega t dt = \frac{V_0 I_0}{2} \left[t - \frac{1}{2\omega} \sin 2\omega t \right]_0^T = \frac{V_0 I_0}{2}$$

For capacitors and inductors,

$$P = \frac{V_0 I_0}{T} \int_0^T \cos \omega t \sin \omega t dt = V_0 I_0 \left[\frac{1}{2\omega} \sin^2 \omega t \right]_0^T = 0 !!!$$



The result for capacitors and inductors is particularly interesting: **on average, they take no net power from the source of energy.** Physically, what happens is that the energy source for the circuit (the power supply) puts energy into C's and L's for half the cycle, while the C's and L's give it back during the other half-cycle. One way to see this behavior is to examine the waveform for C or L shown in the figure. During the 1st and 3rd quarter cycles the signs of V and I are the same, and the signs are opposite during the other two. Thus, energy flows one way during quarter-cycles 1 and 3, while it flows the other during 2 and 4.

The result for resistors forms the basis of the **rms** (root-mean-square) values often used in describing AC voltage and current. Specifically, average power is exactly what a resistor experiences for DC; using $V = IR$ and setting P_{av} for AC equal to the power for DC, we see:

$$P_{DC} = P_{av} = I_{DC}^2 R = \frac{I_0^2 R}{2} = I_{rms}^2 R = P_{rms} \Rightarrow I_{rms} = \frac{I_0}{\sqrt{2}},$$

similarly,

$$P_{DC} = P_{av} = \frac{V_{DC}^2}{R} = \frac{V_0^2}{2R} = \frac{V_{rms}^2}{R} = P_{rms} \Rightarrow V_{rms} = \frac{V_0}{\sqrt{2}}$$

An example of the extent to which **rms** values are used is the description of AC outlet voltage supplied by Georgia Power: 120V is really **120V_{rms}**. Since V_{rms} is less than V_0 (about 70%), $V_0 = 150V$, approximately! Also, the power specifications of light bulbs, for example, is given in average (or rms) power: 40W for a bulb means that **40W = V_{rms} x I_{rms}**.

(Further, since a bulb is essentially a resistor, its resistance under operating conditions can be estimated according to $R = P / I^2$; or, more conveniently since the operating voltage is the usually given parameter, $R = V^2 / P$.) Note also that $|Z| = V_0 / I_0 = V_{\text{rms}} / I_{\text{rms}}$.

Power in a General Impedance: We've seen that the power in individual components can range from 0 (capacitors and inductors) to $\frac{1}{2}V_0I_0$ (resistors). The obvious question is "What about the general case of a Z which is not a single component?" This can be addressed by returning to the integrals above for the general case of ϕ neither 0 nor 90° . Without the details, the result is

$$P_{\text{av}} = \frac{V_0 I_0}{2} \cos\phi = V_{\text{rms}} I_{\text{rms}} \cos\phi.$$

Since ϕ is the phase angle between $V(t)$ and $I(t)$ for the impedance, and ϕ is also related to the impedance according to

$$|Z| = \frac{V_0}{I_0}, Z = \text{Re}(Z) + j\text{Im}(Z) = |Z| @ \phi \Rightarrow \cos\phi = \frac{\text{Re}(Z)}{|Z|} = \text{Re}(Z) \frac{I_0}{V_0}.$$

thus,

$$P_{\text{av}} = P_{\text{rms}} = \frac{I_0^2}{2} \text{Re}(Z).$$

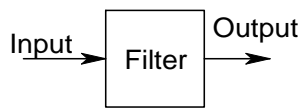
This of course makes sense because the real part of Z is the **resistive** component while the imaginary part is **reactive**. We saw above in the discussion of capacitors and inductors that no power is dissipated by reactive elements.

SUMMARY: AC power is generally expressed as P_{rms} or "equivalent DC power." For this purpose, power is $P_{\text{rms}} = V_{\text{rms}} I_{\text{rms}} \cos\phi$, where ϕ is the "angle" associated with the complex impedance $Z = V/I$. (Cos ϕ is called the "power factor" by electric utility companies.)

Applications of AC Circuits:

Circuit descriptions and analyses are used in two general ways. One is to describe and predict the behavior of existing devices. This of course is the concept of a model for the circuit. The second way is to use the behavior of circuit elements to design circuits to accomplish specific functions. Associated with this objective is that of modifying (or engineering) existing devices to provide improved performance or to meet different objectives.

Obviously, AC circuits may be used in many ways. We will examine the specific application



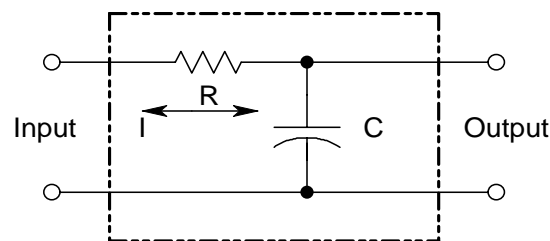
of filtering on the basis of frequency. The concept of filtering can be illustrated as in the inset. An everyday example of filtering in general is the paper used in coffee makers: its purpose is to separate the liquid coffee from the coffee grounds.

Everyday examples of frequency-based filtering are radio and TV tuners, the “equalizer” on stereo systems, and the “crossover” network for stereo speaker systems. In these, the complete signals are superpositions of signals of different amplitudes at different frequencies, and the objective is to sort them out. Tuners, for example, need to “select” from the mixture the frequency of the broadcast chosen; the equalizer separates the different frequency ranges of an audio signal to permit emphasis / de-emphasis of that component only (followed by re-combination); the crossover network is designed to direct the low frequency portion of the audio signal to the “woofers,” the middle range of frequencies to the mid-range speakers, and the high frequency component to the “tweeters.”

RC Low-Pass Filter:

There are many sophisticated methods for creating frequency-based electronic filters. However, the frequency dependence of capacitors and inductors make it possible to obtain filtering behavior also with very simple circuits. For example, consider the simple RC circuit shown below:

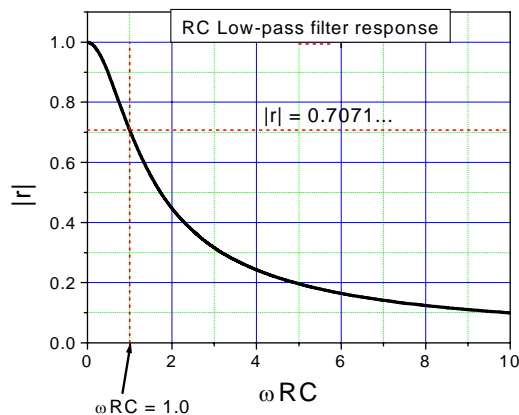
For a voltage V_{in} applied to the input, the output voltage will be:



$$V_{out} = IZ_C = \left(\frac{V_{in}}{Z_R + Z_C} \right) Z_C = \left(\frac{Z_C}{Z_R + Z_C} \right) V_{in}$$

$$r = \frac{V_{out}}{V_{in}} = \frac{Z_C}{Z_R + Z_C} = \frac{(-j/\omega C)}{(R - j/\omega C)} = \frac{1}{j\omega RC + 1}$$

$$|r| = \left| \frac{1}{j\omega RC + 1} \right| = \frac{1}{\sqrt{(\omega RC)^2 + 1}}$$

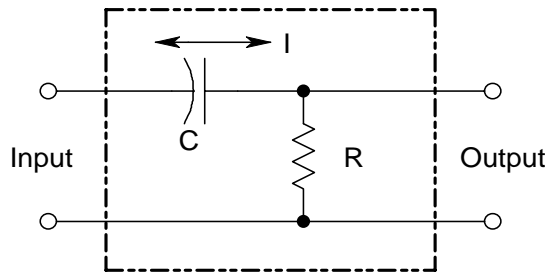


Note that $|r|$ is simply the magnitude of the output to input voltage ratio. The frequency dependence of $|r|$ is shown in the graph.

Obviously, the output is a greater fraction of the input for cases where ωRC is “low.” (In particular, when $\omega RC \leq 1$, $|r| \geq 1 / \sqrt{2}$ and when $\omega RC \geq 10$, $|r| \leq 0.1$. We’ll emphasize this point later.) The emphasis of the circuit’s behavior on “low” frequencies leads to the name “**low-pass**” filter.

RC High-Pass Filter:

Swapping positions of the R and C gives the circuit shown below. Analysis like that used for the low-pass circuit above gives the results:

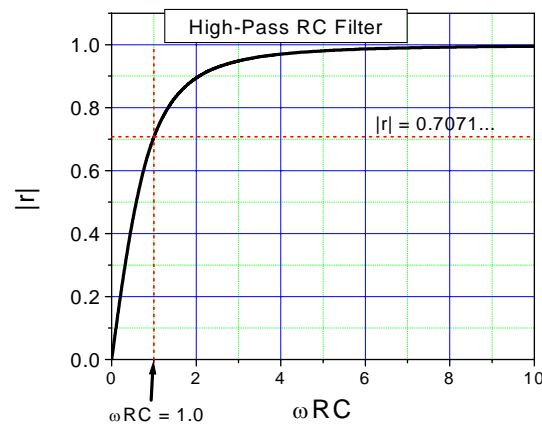


$$r = \frac{V_{out}}{V_{in}} = \frac{Z_R}{Z_R + Z_C} = \frac{j\omega RC}{j\omega RC + 1} = \frac{1}{1 - j/\omega RC}$$

or,

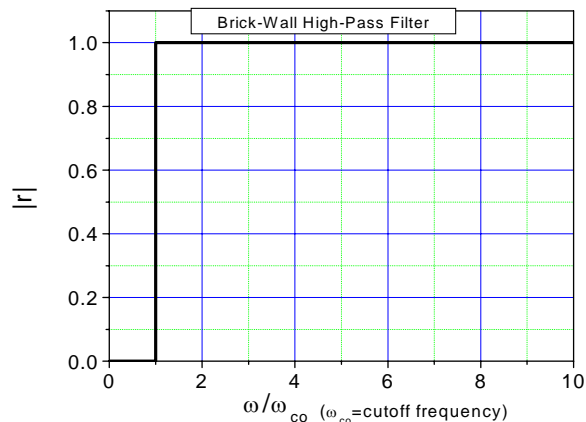
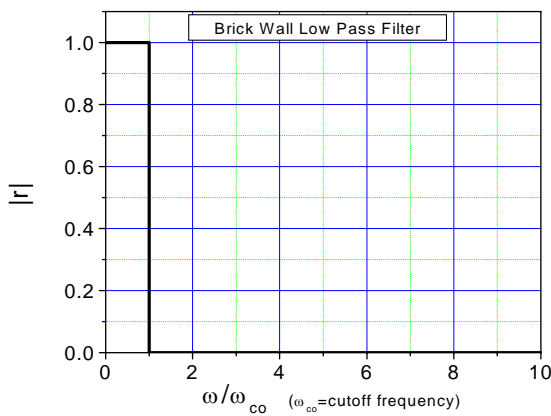
$$|r| = \left| \frac{V_{out}}{V_{in}} \right| = \frac{\omega RC}{\sqrt{(\omega RC)^2 + 1}} = \frac{1}{\sqrt{1 + (1/\omega RC)^2}}$$

This relation is shown graphically on the right. Obviously, the frequency response of this circuit emphasizes “high” over “low” frequencies. Thus, it is a **high-pass** circuit.



Usefulness of “ $\omega RC = 1$ ”:

While these simple RC filters exhibit low-pass and high-pass behavior, they are not perfect filters. “Perfect” filters (which don’t really exist), exhibit the so-called “brick-wall” behavior illustrated in the graphs below.



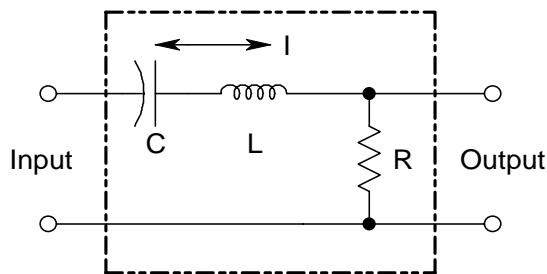
Brick-wall filters have the property of passing at 100% the signals on one side of the **cutoff** frequency and rejecting at 100% everything on the other side. Obviously, our simple RC filters do not have this characteristic. So, how can we characterize the “**cutoff**” characteristics of the RC filters? A reasonable (but imperfect!) cutoff frequency is ω such that $\omega RC = 1$. Referring to the graphs above for the RC versions, we see that most of the rejection occurs for frequencies on one side of this frequency and the best “passing” occurs

for frequencies on the other side. Thus, it is customary to define for the RC circuits shown the **cutoff frequency** $\omega_{co} = \frac{1}{RC}$. Moreover, at that frequency, $|r| = \frac{1}{\sqrt{2}} = 0.7071\dots$

Band-Pass Filters:

Of the “everyday” examples mentioned earlier, tuners (and equalizers) require selecting a specific frequency range from the overall multi-frequency signal. This is the behavior of a **band-pass** circuit. The band-pass behavior can also be achieved with simple RLC circuits as follows:

For this circuit,



With the definitions

$$\omega_0 = \sqrt{\frac{1}{LC}} \text{ and}$$

$$Q = \frac{\omega_0 L}{R},$$

$$|r| = \frac{1}{\sqrt{1 + Q^2 \frac{\omega^2}{\omega_0^2} \left(1 - \frac{\omega_0^2}{\omega^2}\right)^2}}$$

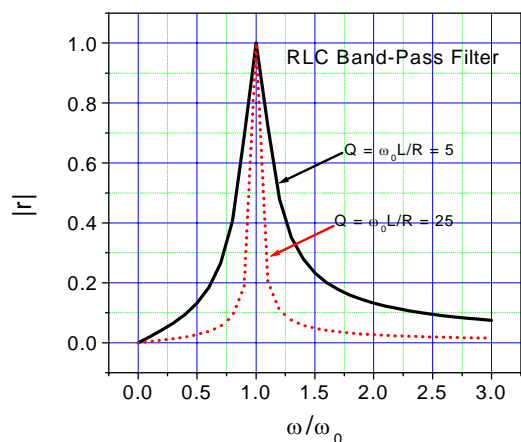
The frequency dependence of $|r|$ is shown in the figure for two different values of Q . Note that the higher the value of Q , the narrower the pass band; that is, for higher Q , the circuit is more “selective.”

$$r = \frac{V_{out}}{V_{in}} = \frac{Z_R}{Z_R + Z_L + Z_C} = \frac{R}{R + j(\omega L - 1/\omega C)}$$

and,

$$|r| = \left| \frac{Z_R}{Z_R + Z_L + Z_C} \right| = \frac{R}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}$$

$$= \frac{1}{\sqrt{1 + \left(\frac{\omega L}{R}\right)^2 \left(1 - \frac{1}{\omega^2 LC}\right)^2}}$$



AC Problems Focused on Filter Design:

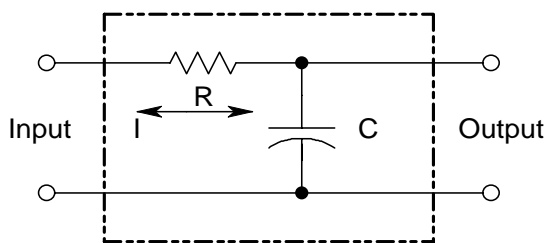
Example 4:

Given a signal, $V = 2.0V @ 100kHz + 1.0V @ 2kHz$, find RC values for the **appropriate** circuit to **reject** at least 90% of the 100 kHz component while **passing** at least 80% of the 2 kHz component.

$$\text{want } \left| \frac{V_{\text{out}}}{V_{\text{in}}} \right| \leq 0.1 @ 100 \text{ kHz} \quad (100\% - 90\% = 10\%) \text{ Summary:}$$

$$\text{and } \left| \frac{V_{\text{out}}}{V_{\text{in}}} \right| \geq 0.8 @ 2 \text{ kHz}$$

- Rejection of **higher** frequency and passing of **lower** requires a **low-pass** filter:



$$V_{\text{out}} = IZ_C = \left(\frac{V_{\text{in}}}{Z_R + Z_C} \right) Z_C = \left(\frac{Z_C}{Z_R + Z_C} \right) V_{\text{in}}$$

$$r = \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{Z_C}{Z_R + Z_C} = \frac{(-j/\omega C)}{(R - j/\omega C)} = \frac{1}{j\omega RC + 1}$$

$$|r| = \left| \frac{1}{j\omega RC + 1} \right| = \frac{1}{\sqrt{(\omega RC)^2 + 1}}$$

$$|r| = \frac{1}{\sqrt{1 + (\omega RC)^2}} \Rightarrow RC = \frac{1}{|r|\omega} \sqrt{1 - |r|^2}$$

$$@ 100 \text{ kHz}, \quad |r| \leq 0.1 \Rightarrow RC \geq \frac{1}{2\pi \times 10^5 (0.1)} \sqrt{1 - (0.1)^2} \Omega F$$

$$RC \geq \frac{10^{-4}}{2\pi} \approx 1.6 \times 10^{-5} \Omega F$$

$$@ 2 \text{ kHz}, \quad |r| \geq 0.8 \Rightarrow RC \leq \frac{1}{(2\pi)1.6 \times 10^3} \sqrt{1 - (0.8)^2} \Omega F$$

$$RC \leq 6.0 \times 10^{-5} \Omega F$$

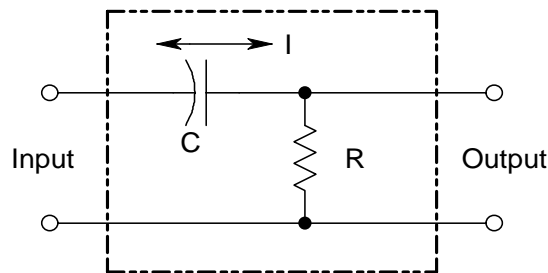
$$\text{Thus, } 1.6 \times 10^{-5} \Omega F \leq RC \leq 6.0 \times 10^{-5} \Omega F$$

Example 5:

For the same signal as **example 4**, find RC values and the appropriate circuit to **reject** 95% or more of the 2 kHz component while **passing** 90% or more of the 100 kHz component.

Summary: want $|r| \geq 0.90$ @ 100 kHz, and $|r| \leq 0.05$ @ 2 kHz.

- Rejecting **low frequencies** while passing the **high frequencies** requires a **high-pass** circuit:



$$r = \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{Z_R}{Z_R + Z_C} = \frac{j\omega RC}{j\omega RC + 1} = \frac{1}{1 - j/\omega RC}$$

or,

$$|r| = \left| \frac{V_{\text{out}}}{V_{\text{in}}} \right| = \frac{\omega RC}{\sqrt{(\omega RC)^2 + 1}} = \frac{1}{\sqrt{1 + (1/\omega RC)^2}}$$

$$|r| = \frac{1}{\sqrt{1 + (1/\omega RC)^2}} \Rightarrow RC = \frac{|r|}{\omega} \frac{1}{\sqrt{1 - |r|^2}}$$

$$\text{@ 100 kHz, } RC \geq 3.3 \times 10^{-6} \text{ } \Omega\text{F}$$

$$\text{@ 2 kHz, } RC \leq 4.0 \times 10^{-6} \text{ } \Omega\text{F}$$

$$\therefore \mathbf{3.3 \times 10^{-6} \text{ } \Omega\text{F} \leq RC \leq 4.0 \times 10^{-6} \text{ } \Omega\text{F}}$$

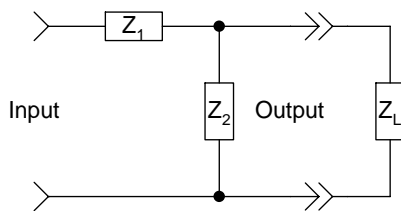
NOTES:

1. For both examples, the result was the **product** of R and C (in ohms and Farads). Obviously, there are infinitely many combinations which will give the same RC product; thus the result does not yield specific values for R and C. This provides the opportunity to meet at least one other performance objective which may involve R and / or C, a typical one being the sensitivity of the filter's performance to **loading**. (Otherwise, R and C can be chosen by convenience as long as their product is within the required RC range.)

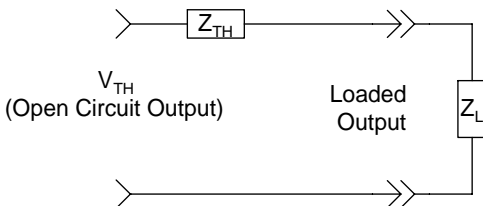
2. Also, it is possible for the low and high frequency design targets to be incompatible. For example, changing the high frequency objective for **example 5** from "passing 90% or more" to "passing 95% or more" leads to non-overlapping values for the RC product for the two frequencies. (Check it out!)

Effect of “Loading” on Filter Performance:

In operation, filter circuits are connected between the source of the signal to be filtered and the device, or **load**, to which the signal is to be transferred. However, the filter circuits above were designed and analyzed with no load attached or considered. Practical considerations, therefore, require analysis of the possible effect of connecting an actual load. In other words, under what conditions might the characteristics of the load bring the filter’s performance outside the range specified by the design criteria?



The effect of loading is most directly analyzed by use of Thevenin’s theorem. From that perspective, dividing the circuit with load at the output terminals indicated, the Thevenin voltage is simply the output voltage derived before for the “unloaded” circuit, while the Thevenin impedance is the parallel equivalent of Z_1 and Z_2 .



$$V_{TH} = V_{in} \frac{Z_2}{Z_1 + Z_2} = V_{in} r$$

$$Z_{TH} = Z_1 // Z_2 = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

(see previous analysis for meaning of "r")

From these expressions and analysis of the “Thevenin-equivalent” version of the loaded circuit, the result for the “loaded” output is:

$$V_{out} = V_{load} = \left(\frac{V_{TH}}{Z_{TH} + Z_L} \right) Z_L = V_{in} r_o \left(\frac{Z_L}{Z_{TH} + Z_L} \right), \quad r_o = \text{"open-circuit" } r$$

$$r_{load} = \frac{V_{load}}{V_{in}} = r_o \left(\frac{Z_L}{Z_{TH} + Z_L} \right) = r_o \left(\frac{1}{\frac{Z_{TH}}{Z_L} + 1} \right) = r_o, \quad Z_{TH} \ll Z_L$$

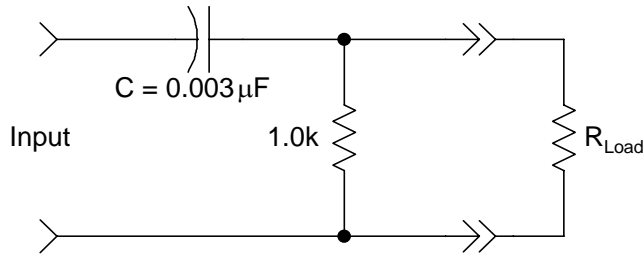
In summary, if $Z_{TH} \ll Z_L$, $r_{load} \sim r_o$. For RC filters, note that Z_1 or Z_2 will be an R and the other a C. Thus, regardless of whether the filter is low- or high-pass, the Thevenin equivalent impedance is:

$$Z_{TH} = \frac{R(-j/\omega C)}{(R - j/\omega C)} = R \frac{1}{(j\omega RC + 1)}$$

$$|Z_{TH}| = R \sqrt{\frac{1}{(\omega RC)^2 + 1}} \leq R \text{ for all } \omega$$

Example 6:

The RC high-pass circuit below was designed to meet the filtering specifications:



$$|r| \geq 0.7 \text{ @ } 100 \text{ kHz and}$$

$$|r| \leq 0.1 \text{ @ } 500 \text{ Hz.}$$

Calculate the lowest value of R_{Load} (load resistance) for which the circuit will still meet the specifications.

One straightforward way of proceeding is to recognize that the load resistance is in parallel with the 1.0k resistor of the filter. The overall filtering effect will therefore be determined by the capacitor and the parallel equivalent of R_{load} and 1.0k. The question then becomes that of finding the value of R_{eq} that will bring $|r|$ to 0.7. R_{load} can then be calculated from R_{eq} since

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_{\text{load}}} + \frac{1}{1.0\text{k}} \Rightarrow \frac{1}{R_{\text{load}}} = \frac{1}{R_{\text{eq}}} - \frac{1}{1.0\text{k}}.$$

From before,

$$R_{\text{eq}}C = \frac{|r|}{\omega} \sqrt{\frac{1}{1 - |r|^2}} = \frac{0.7}{2\pi \times 10^5} \sqrt{\frac{1}{1 - (0.7)^2}} = 1.56 \times 10^{-6}$$

$$R_{\text{eq}} = \frac{1.56 \times 10^{-6}}{3 \times 10^{-9}} = 520\Omega$$

$$\frac{1}{R_{\text{load}}} = \frac{1}{520} - \frac{1}{1000} \cong \frac{1}{1083}$$

Therefore, for load resistances of values $R_{\text{load}} < 1083\Omega$, the filtering behavior of this circuit no longer meets the specified requirements.