P2211K

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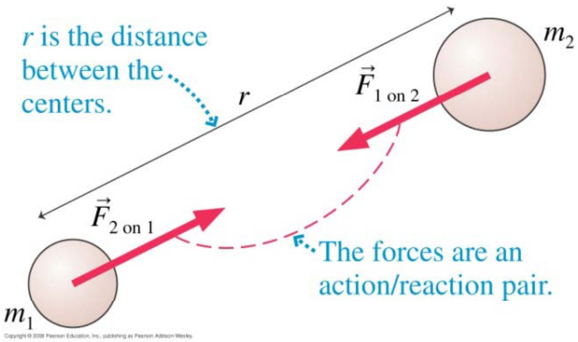
Newton's law of gravity, gravitational potential energy, and "escape speed":

- Newton's "law" of gravity, as introduced in chapter 6, is:



$$F_{1\,on\,2} = F_{2\,on\,1} = G\frac{m_1 m_2}{r^2}$$

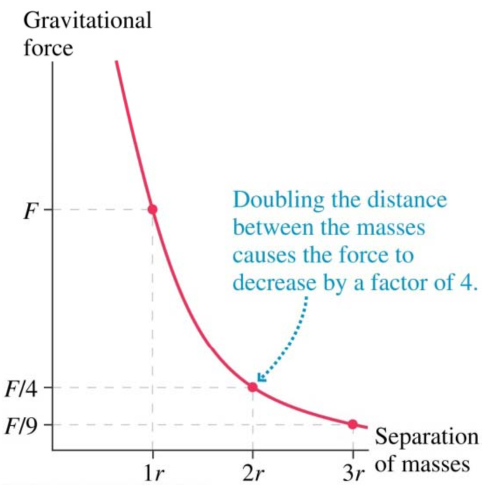
- For an object of mass ***m*** acted upon by the earth's gravitational field, this becomes:

$$F_g = \left(G\frac{M_E}{r^2}\right)m = mg(r),$$

At the surface of the earth, this becomes

$$F_g = \left(G\frac{M_E}{R_E^2}\right)m = mg(R_E\,),\ where$$

$$g(R_E\,) = 9.8\ m/s^2$$



The concept of a "field":

- The expression for Newton's law of gravitation can be reorganized to focus on m_1 as the "source" and m_2 as the "recipient" of the effect:

$$F_{1\,on\,2} = \left(G\frac{m_1}{r^2}\right)m_2 = Z_1(r)m_2$$

- In this arrangement, $Z_1(r)$ is the "effect" generated by m_1 that leads to the force experienced by m_2.
- Obviously, if m_2 is moved to other locations, it would "feel" a force due to m_1 at all of them as described by the force law. Thus, m_1 sets up an effect that exists at all space positions regardless of whether or not there is a "recipient" mass to experience the force it creates. This "effect," described by $Z_1(r)$, is the ***gravitational field*** associated with m_1. From this perspective, the force on m_2 is the result of its interaction with the ***gravitational field*** at its location.
- Just as obviously, the ***field*** concept also applies to m_2: it sets up a gravitational field and m_1 experiences a force as a result of its interaction with m_2***'s*** field.

Questions:

- If m_1 "wiggles" what is the effect on m_2, and how long does it take for m_2 to "know"?

Gravitational potential energy (and "escape" speed)**:**

Question: What minimum initial speed at the surface of the earth is necessary for a projectile to "escape" the earth's gravitational field?

- Basic Idea: the ***minimum escape speed*** is that which will lead to $v_f = 0$ ***as*** $r \to \infty$.
- Speed suggests kinetic energy, and we already know about gravitational potential energy near the surface of the earth. Thus, if we can develop a more general description of the earth's gravitational potential energy, we can use conservation of energy methods to find the minimum v_{escape}.
- From the discussion in section 13.5 (p. 394), we learn that the expression for gravitational potential energy is

$$U_g(r) = -G\frac{M_E}{r}m$$

- Note that r is in the denominator so that U_g ***goes to 0 as r goes to infinity***. Thus, the final values for both the kinetic and potential energies is zero, meaning that the total energy at the end is zero: $E_{Tf} = K_f + U_f = 0$.
- Therefore, from the conservation of energy principle that $E_{Tf} = E_{Ti}$, we conclude that $E_{Ti} = K_i + U_i = 0$ and that $K_i = -U_i$. ***Re example 13.2 (p. 396)***, the result is that

$$\tfrac{1}{2}mv_{escape}^2 = G\frac{M_E}{R_E}m \Rightarrow v_{escape}^2 = 2\left(G\frac{M_E}{R_E}\right)\left(\frac{R_E}{R_E}\right) = 2\left(G\frac{M_E}{R_E^2}\right)R_E = 2gR_E$$

Thus,

$$v_{escape} = \sqrt{2gR_E} = 11.174\times 10^3 \text{ m/s} \simeq 22{,}000 \text{ mi/hr}$$

Gravitational potential energy near the earth's surface:

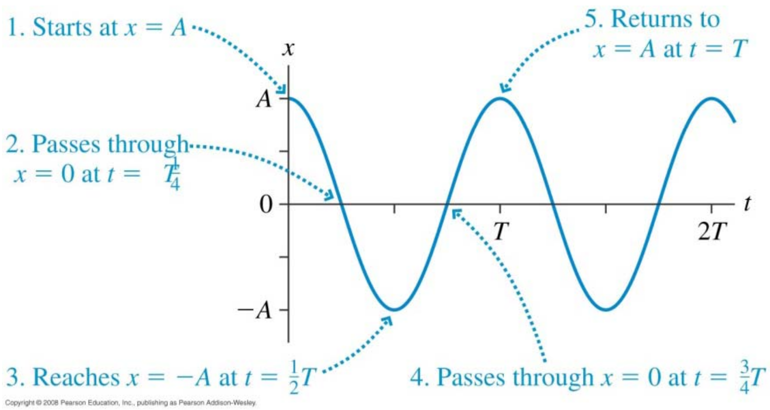
- When we first introduced it, we described gravitational potential energy as $\boldsymbol{U_g = mgh}$. So, how is this compatible with the more general description above?
- **Approach:** Using the more general relation, consider the difference in potential energy for a position at the earth's surface and one h (<< R_E) above it:

$$
\begin{aligned}
U_g(R_E + h) - U_g(R_E) &= -G\frac{M_E}{R_E + h}m + G\frac{M_E}{R_E}m \\
&= GM_E\left(\frac{1}{R_E} - \frac{1}{R_E + h}\right)m \\
&= GM_E\left[\frac{R_E + h - R_E}{R_E\left(R_E + h\right)}\right]m \\
&= GM_E\left[\frac{h}{R_E\left(R_E + h\right)}\right]m \\
&\simeq h\left[\frac{GM_E}{R_E^2}\right]m,\ for\ R_E >> h \\
&= mgh
\end{aligned}
$$

- Thus, for the case where the distance from the earth's surface is small compared to its radius, the more general version reduces to the simpler one we used previously.

Oscillations, vibrations, and simple harmonic motion (SHM)

Definition of SHM: Sinusoidal (or cosine) oscillations



Functional form: $x(t) = A\cos(\omega t + \varphi)$, where

- $x(t)$ = position
- ω = angular frequency (radians / s) = $2\pi f$
- f = frequency in Hertz (Hz)
- A = amplitude
- φ = phase (to come later)

Other relations:

- f = frequency = repetitions / time
- T = Period = time / repetition
- T = 1/f

- Note that the motion has gone through one complete cycle when $\omega t = 2\pi$. Thus the functional form can be rewritten in many forms:

$$x(t) = A\cos(\omega t + \phi) = A\cos(2\pi ft + \phi) = A\cos(\frac{2\pi t}{T} + \phi)$$

- Also, note that velocity and acceleration are related to position by

$$x(t) = A\cos(\omega t + \phi)$$

$$v(t) = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi)$$

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t + \phi)$$

So how do we get SHM?

- ***Basic answer***: spring-like forces, *i.e.*, ones described by a linear restoring behavior--- F = -kx.
- ***Why is this important***?
  1. SHM gives the most basic version of a ***periodic*** motion;
  2. The characteristic of a linear restoring force is a good approximation of more complex forces (one example is the simple pendulum);
  3. Chemical bonds, and thus their vibration behaviors, can be approximated by this type of force;
  4. The deflection (or bending) behavior of structural units (beams, etc.) can be described to a good approximation by these forces;
  5. etc.
- How do we know spring-like forces produce SHM? Look at the "F = ma" relation:

$$ma(t) = -kx(t) \Rightarrow m\frac{d^2x(t)}{dt} = -kx(t) \Rightarrow \frac{d^2x(t)}{dt} + \left(\frac{k}{m}\right)x(t) = 0$$

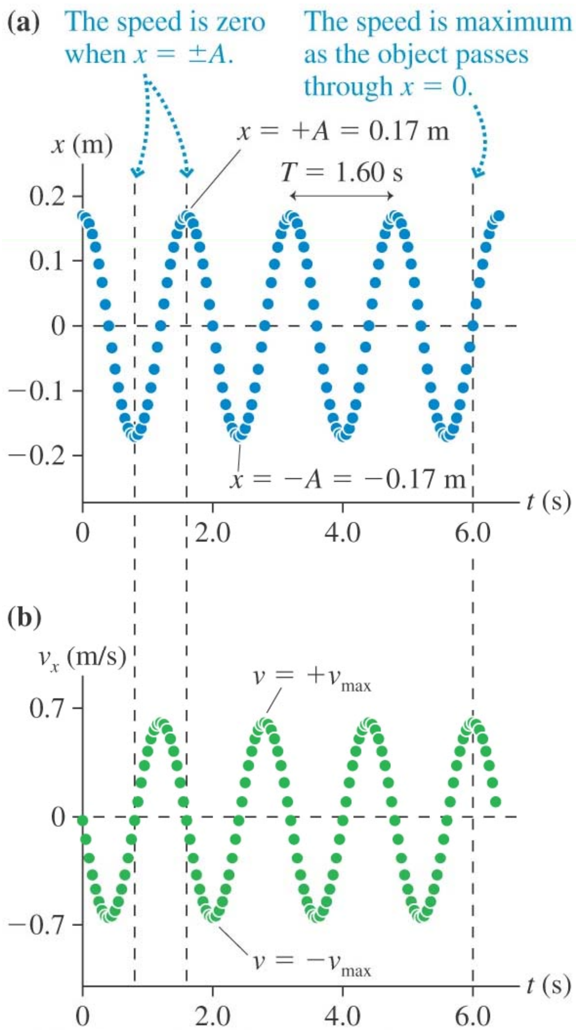
For this type of differential equation, the general solution is

$$x(t) = A\sin(\omega t) + B\cos(\omega t) \qquad \text{with } \omega^2 = \frac{k}{m}$$

(A and B are amplitude parameters determined by the initial conditions.)

- Thus, this demonstrates that SHM is the result of the linear restoring force.

Dynamics of SHM



$$x(t) = A\cos(\omega t + \phi)$$

$$v(t) = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi)$$

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t + \phi)$$

- From the relations for ***x(t), v(t), and a(t),*** we see that the ***maximum displacement is A***, ***the maximum speed is ωA***, and ***the maximum acceleration is ω²A***. Furthermore, from the relations between sin and cos functions, ***the maximum displacement occurs when the speed is zero, and the maximum speed occurs when the displacement is zero***.
- Also, we know from previous discussions that the spring-mass system is ***conservative*** so that

$$\Delta K + \Delta U = 0 \Rightarrow E_T = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2$$

and that

$$E_T = \tfrac{1}{2}mx_{max}^2 = \tfrac{1}{2}kA^2$$

or

$$E_T = \tfrac{1}{2}mv_{max}^2 = \tfrac{1}{2}m\omega^2 A^2$$

Problem 14-13. A 200 g mass attached to a horizontal spring oscillates at a frequency of 2.0 Hz. At $t = 0$, the mass is at x = 5.0 cm and has $v_x = -30.0$ cm/s. Determine:

a. The period. $T = 1/f = 0.5$ sec

b. The angular frequency. $\omega = 2\pi f = 12.57$ rad/s

c. The amplitude. $E_T = ½kA^2$; Need $k = m\omega^2 = 31.6$ N/m &
$E_T = ½kx^2 + ½mv^2 = 0.0485$ J, so
$A = (x^2 + mv^2/k)^{½} = (x^2 + v^2/\omega^2)^{½} = 5.54$ cm

d. The phase constant. $x(t) = A\cos(\omega t + \varphi)$, so $x(0) = A\cos(\varphi)$ &
$\varphi = \cos^{-1}[x(0)/A] = \cos^{-1}[5/5.54] = 25.5° = 0.445$ rad

e. The maximum speed. $E_T = ½m(v_{max})^2$, so $v_{max} = (2E_T/m)^{½} = 69.6$ cm/s
also, $v_{max} = A\,\omega$

f. The maximum acceleration. $a_{max} = \omega^2 A = 875.3$ cm/s^2

g. The total energy. $E_T = 0.0485$ J

h. The position at $t = 0.40$ s. $x(0.40) = 5.54\cos(12.57 * 0.04 + 0.445)$ cm
$= 3.23$ cm

Assignment:

- **Continue reading and working on Chapter 12; especially you should review the book's worked-out examples.**
- **Begin reading Chapter 13 with a focus on the more general form for gravitational potential energy and the "escape velocity," section 13-5.**