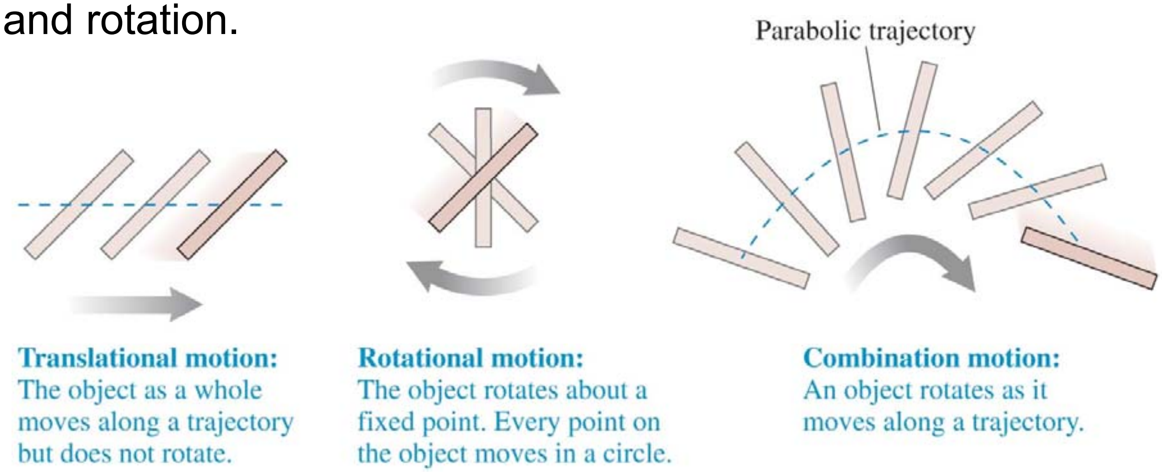
P2211K

10 / 26 / 2010

Chapter 12: A more realistic view of objects and their motion—rigid bodies

- Objects have extent—i.e., they are not simply points
- In general, the motion of real objects is a combination of translation (our focus up to now) and rotation.



- To describe this motion, we need to introduce several new concepts: ***center of mass, moment of inertia (or rotational inertia), torque, and angular momentum***;
- We will describe ***rigid-body dynamics*** from two perspectives: ***work-energy and torque*** (obviously, these are inter-related and equivalent from a technical standpoint);
- We will work with ***rigid-body equilibrium*** problems—these need to add ***torque*** to the equilibrium condition;
- We will extend the ***rigid-body dynamics*** discussion to the description of ***rolling motion***;
- We (probably) will introduce the ***cross-product***, which is another way to multiply vectors.

- "Rigid bodies" are objects made up of a collection of "points stuck together" in such a way that that "Rigid bodies" are objects made up of a collection of "points stuck together" in such a way that the object's size and shape do not change as it moves—think a steel disk, etc. (Obviously, even a steel disk can vibrate as it moves and many objects are "soft" bodies, but their description adds complexity that must wait for another day.)

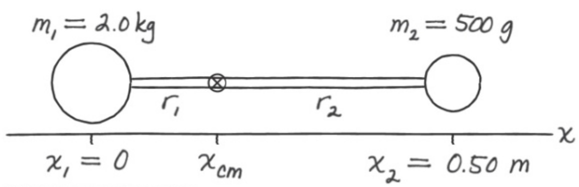
So how do we describe the kinematics and dynamics of rigid bodies? Obviously, by considering both translation and rotation, but the 1st question is how can we simplify this when the stuck-together pieces are moving in what appears to be a complicated way?

- 1st idea: the combined motion can be described a translation of the ***center of mass*** plus rotation about the ***center of mass***.
- So, what is the ***center of mass***? This basically is the idea of a "balance point," which probably is familiar to you from your playground days and see-saws.
- 2nd idea: we need to know how mass enters into the ***dynamics of rotation*** (energy, work, causes, etc.), and this leads to the idea of ***rotational inertia***.
- So, what is ***rotational inertia (or moment of inertia)***? We'll see, but it has to do with ***the amount of mass and how it is distributed*** relative to the center of mass (or axis of rotation).

Center of Mass:

- For example, in the figure, M_1 is 4 times as massive as M_2, so you know instinctively that the balance point needs to be closer to M_1 than M_2. (This intuition also includes the concept of torque & equilibrium that we'll consider before we finish this chapter.) In fact, we know that x_{cm} can be calculated by the relation

$$(2.0\text{ kg})\, x_{cm} = (0.5\text{ kg})(0.5\text{m} - x_{cm}) \text{ so that } x_{cm} = 0.1\text{ m}$$



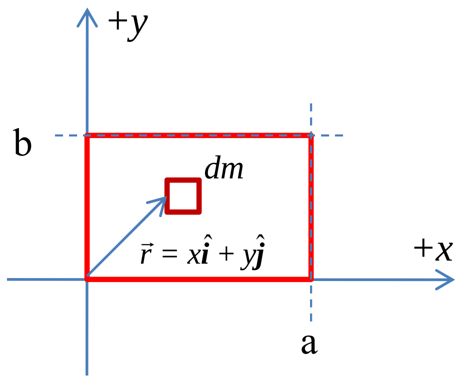
- This two-element system suggests the more general approach for a system with a larger number of elements lying in a plane (but not a straight line). This is given by the book's equation 12-4, shown below.

$$x_{cm} = \frac{1}{M}\sum_i m_i x_i = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

$$y_{cm} = \frac{1}{M}\sum_i m_i y_i = \frac{m_1y_1 + m_2y_2 + m_3y_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

Actually, rigid bodies are better described as continuous objects rather than a collection of discrete mass points. Doing so translates the summation of eq. 12-4 to the integration expressed by eq. 12-5. In 12-5's separate x_{cm} and y_{cm} expressions are combined into a single vector expression for r_{cm}, the result is as expressed below:

$$x_{\mathrm{cm}} = \frac{1}{M}\int x\,dm \quad \text{and} \quad y_{\mathrm{cm}} = \frac{1}{M}\int y\,dm \quad \Longrightarrow \quad \vec{r}_{cm} = \frac{1}{M}\int_{object} \vec{r}dm$$



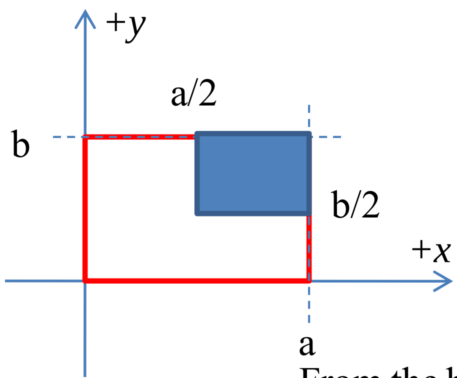
- Consider, for example, the rectangular shape of dimensions a×b and total mass M.
- In the integrals, dm is a differential mass element (an amount approaching zero without being zero) easily related to the total mass of the object (M) and its total area (Area), thus, for uniformly distributed mass, the mass / area is the same for all pieces of the object ranging from the whole thing down to a differential element of area.
- Thus, the mass in a differential area element of dimensions dx by dy is

$$\frac{dm}{dArea} = \frac{M}{ab} = \frac{dm}{dxdy} \Rightarrow dm = \left(\frac{M}{ab}\right)dxdy$$

- With this expression for dm and its relation to the origin as given by $\vec{\boldsymbol{r}}$, the integral for the whole object becomes:

$$
\begin{aligned}
\vec{r}_{cm} &= \frac{1}{M}\int_{object}\vec{r}dm \\
&= \frac{1}{M}\iint_{x\&y}\left(x\hat{\boldsymbol{i}}+y\hat{\boldsymbol{j}}\right)\left(\frac{M}{ab}\right)dxdy \\
&= \left(\frac{1}{ab}\right)\left[\int_{y=0}^{y=b}dy\int_{x=0}^{x=a}(x\hat{\boldsymbol{i}})dx+\int_{x=0}^{x=a}dx\int_{y=0}^{y=b}(y\hat{\boldsymbol{j}})dy\right] \\
&= \left(\frac{1}{ab}\right)\left[b\left(\frac{a^2}{2}\hat{\boldsymbol{i}}\right)+a\left(\frac{b^2}{2}\hat{\boldsymbol{j}}\right)\right]=\left(\frac{a}{2}\hat{\boldsymbol{i}}+\frac{b}{2}\hat{\boldsymbol{j}}\right)
\end{aligned}
$$

- So, the position of the cm for this symmetrical object with uniformly-distributed mass is at its geometric midpoint---big surprise!!!
- (By the way, the double integrals above are equivalent to the book's eq. 12-5 if their dm for the x-integral is that of a strip parallel to the y axis of width dx and length b; for the y-integral, the strip is parallel to x with width dy and length a.)



- Now, if we cut a chunk out of the rectangular piece, what will be the ***cm*** of the remainder?
- If the piece with the chunk in place is regular ,and if the chunk also is regular, then this is most easily accomplished by taking advantage of the fact that operations for calculating the cm are additions (an integral is an addition); specifically:

From the book's eq. 12 - 5, the cm of the two pieces as a unit is :

$$\vec{r}_{cm}\Big|_{M1+M2} = \frac{M1\vec{r}_{cm}\big|_{M1} + M2\vec{r}_{cm}\big|_{M2}}{M1+M2}$$

thus, from algebraic manipulation, the cm of the piece with the missing chunk is :

$$\vec{r}_{cm}\Big|_{M1} = \frac{(M1+M2)\vec{r}_{cm}\big|_{M1+M2} - M2\vec{r}_{cm}\big|_{M2}}{M1}$$

- For the example shown, the piece cut out is ¼ of the total and is the section at the upper right shaded in blue. In this case, we know the cm for the "whole piece,' and we can easily write the cm of the regular chunk to be removed. Thus:

$$\vec{r}_{cm}\Big|_{M1} = \frac{(M1+M2)\vec{r}_{cm}\big|_{M1+M2} - M2\vec{r}_{cm}\big|_{M2}}{M1}$$

$$\mathrm{M1+M2=M,\ M1=\tfrac{3}{4}M,\ \ M2=\tfrac{1}{4}M,}$$

$$\vec{r}_{cm}\Big|_{M1+M2} = \tfrac{1}{2}\left(a\hat{\mathbf{i}} + b\hat{\mathbf{j}}\right) \text{ and } \vec{r}_{cm}\Big|_{M2} = \tfrac{3}{4}\left(a\hat{\mathbf{i}} + b\hat{\mathbf{j}}\right), \text{ so}$$

$$\Rightarrow$$

$$\vec{r}_{cm}\Big|_{M1} = \frac{(M)\vec{r}_{cm}\big|_{M1+M2} - \tfrac{1}{4}M\vec{r}_{cm}\big|_{M2}}{\tfrac{3}{4}M}$$

$$= \frac{(M)\tfrac{1}{2}\left(a\hat{\mathbf{i}} + b\hat{\mathbf{j}}\right) - \left(\tfrac{1}{4}M\right)\tfrac{3}{4}\left(a\hat{\mathbf{i}} + b\hat{\mathbf{j}}\right)}{\tfrac{3}{4}M}$$

$$= \tfrac{5}{12}\left(a\hat{\mathbf{i}} + b\hat{\mathbf{j}}\right)$$

Rotational kinetic energy: $K_{rot} = \sum_{\text{all } i} \frac{1}{2} m_i v_i^2$

Reminder:

angular position: $\theta = \dfrac{\text{arc length}}{\text{radius}}$ (radians)

angular velocity: $\omega = \dfrac{d\theta}{dt}$ (radians/s)

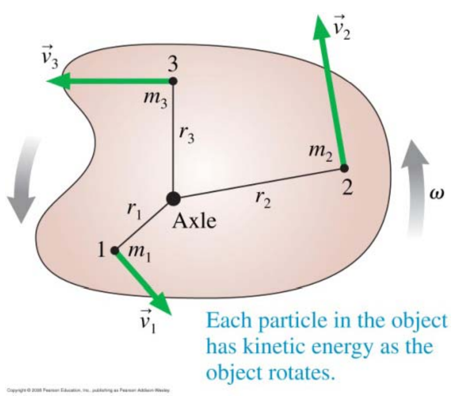
angular acceleration: $\alpha = \dfrac{d\omega}{dt}$ (radians/s^2)

distance = arc length: $s = r\theta$

tangential speed: $v_t = \dfrac{ds}{dt} = r\dfrac{d\theta}{dt} = r\omega$

tangential acceleration: $a_t = \dfrac{dv_t}{dt} = r\dfrac{d\omega}{dt} = r\alpha$

K_{rot} is conveniently related to angular velocity as follows:



$$K_{rot} = \sum_{\text{all } i} \tfrac{1}{2} m_i v_i^2 = \sum_{\text{all } i} \tfrac{1}{2} m_i (r_i\omega)^2 = \tfrac{1}{2}\left(\sum_{\text{all } i} m_i r_i^2\right)\omega^2$$

- Based on this, the ***moment of inertia*** or (***rotational inertia***), symbolized by ***I***, is defined as follows:

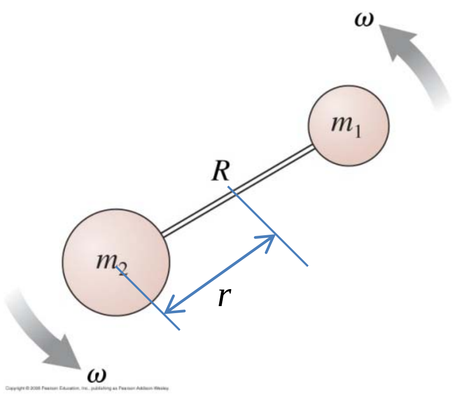
$$I = \sum_{\text{all } i} m_i r_i^2, \text{ so that } K_{rot} = \tfrac{1}{2} I\omega^2$$

- ***Because of the r_i dependence, I depends on the distribution of mass AND on the specific axis of rotation***.

Examples:

a. Find the center of mass (relative to M_2) for the two-mass object;

$$r_{cm} = R\left(\frac{m_1}{m_1 + m_2}\right)$$



b. Find I for rotation about the midpoint (R/2);

$$\boxed{I_{R/2} = \left(m_1 + m_2\right)\frac{R^2}{4}}$$

c. Find I for rotation about M_1;

$$\boxed{I_{m_1} = m_2 R^2}$$

d. Find I for rotation about M_2;

$$\boxed{I_{m_2} = m_1 R^2}$$

e. Find r that gives the minimum I. (Note that this is r_{cm} !!!)

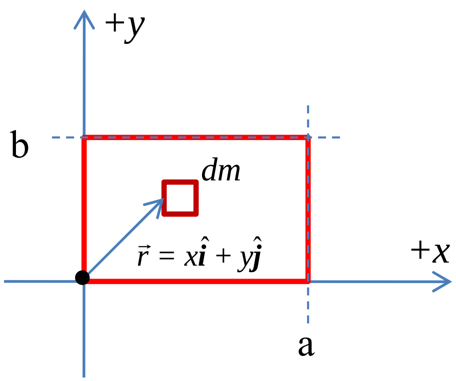
$$\boxed{\begin{aligned} & I = m_1\left(R - r\right)^2 + m_2 r^2 = m_1\left(R^2 - 2Rr + r^2\right) + m_2 r^2 \\ & \frac{dI}{dr} = 0 = m_1\left(-2R + 2r\right) + m_2\left(2r\right) \\ & r\left(m_1 + m_2\right) = Rm_1 \Rightarrow \ r = R\frac{m_1}{\left(m_1 + m_2\right)} \end{aligned}}$$

For a continuous object,

$$I = \sum_{\text{all } i} m_i r_i^2 \Rightarrow I = \int_{\text{object}} r^2 dm$$

where r is the distance from mass element dm to the axis of rotation.

Example: For the uniformly distributed rectangular mass of dimensions a × b shown below, calculate I for rotation about an axis perpendicular to the xy plane and passing through the origin.



$$\begin{aligned}
I_{axis} &= \int_{object} r^2 dm \\
&= \left(\frac{M}{ab}\right) \iint_{x\&y} \left(x^2 + y^2\right) dxdy \\
&= \left(\frac{M}{ab}\right) \left[\int_{y=0}^{y=b} dy \int_{x=0}^{x=a} (x^2)dx + \int_{x=0}^{x=a} dx \int_{y=0}^{y=b} (y^2)dy \right] \\
&= \left(\frac{M}{ab}\right) \left[b\left(\frac{a^3}{3}\right) + a\left(\frac{b^3}{3}\right) \right] = M\left(\frac{a^2}{3} + \frac{b^2}{3}\right)
\end{aligned}$$

Questions *f*or the shape above:

a. What is I for rotation about an axis perpendicular to the xy plane and passing through the center of mass? (Hint: it's the same procedure as above with the limits to the integrals changed so that $-a/2 \le x \le a/2$ and $-b/2 \le y \le b/2$.)

b. What is I for rotation about the x axis?

c. What is I for rotation about the y axis?

Other properties of I:

- ***I is least for rotations*** about an axis passing through the center of mass;
- ***Parallel axis theorem***: for rotation about an axis parallel to one passing through the center of mass,

$$I_{axis} = I_{cm} + MR^{2,}$$

  where M is the total mass of the object and R is the distance between the axis and one through the cm.
- ***Addition and subtraction of I:*** The total I for a collection of objects rotating about the same axis is the sum of the I's for each about the axis:

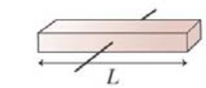
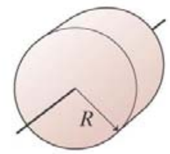
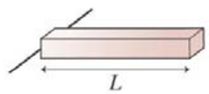
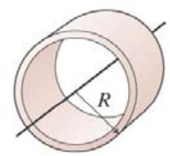
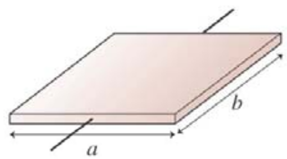
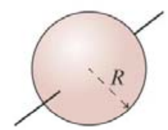
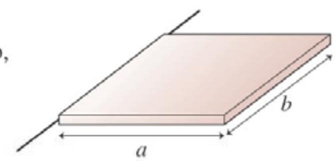
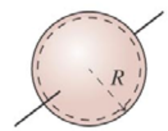
$$I_{axis,T} = I_{axis,1} + I_{axis,2} + I_{axis,3} + \dots$$

- ***Similarly,*** the rotational inertia of an object with a missing piece can be calculated by subtraction. (This is useful when it is easier to calculate $I_{axis,T}$ and $I_{axis,2}$ than to calculate $I_{axis,1}$ directly.)

$$I_{axis,1} = I_{axis,T} - I_{axis,2}$$

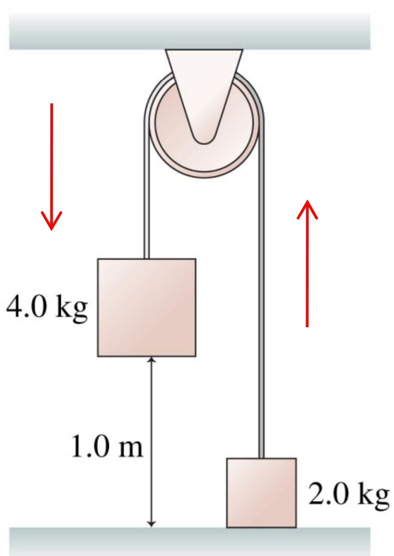
Rotational Inertia of some standard shapes about typical axes (p. 347 in book):

TABLE 12.2 Moments of inertia of objects with uniform density

Object and axis	Picture	I	Object and axis	Picture	I
Thin rod, about center		$\frac{1}{12}ML^2$	Cylinder or disk, about center		$\frac{1}{2}MR^2$
Thin rod, about end		$\frac{1}{3}ML^2$	Cylindrical hoop, about center		MR^2
Plane or slab, about center		$\frac{1}{12}Ma^2$	Solid sphere, about diameter		$\frac{2}{5}MR^2$
Plane or slab, about edge		$\frac{1}{3}Ma^2$	Spherical shell, about diameter		$\frac{2}{3}MR^2$

Prob 12-70 (sort of):The two blocks in the figure are connected by a massless rope that passes over a pulley. The pulley is 12 cm in diameter, has a mass of 2.0 kg, and has rotational inertia like a solid disk. The rope does not slip on the pulley and the pulley rotates on its axle without friction.

- If the blocks are released from rest, what is the speed of the 4.0 kg block as it reaches the floor?
- How long does it take to reach the floor?



- Basic principle: $\Delta K_T = W_T$
- $\Delta K_4 + \Delta K_2 + \Delta K_P = W_{g4} + W_{g2}$
- $\frac{1}{2}M_4v^2 + \frac{1}{2}M_2v^2\ \frac{1}{2}I\omega^2 = M_4gh - M_2gh$
- String doesn't slip $\rightarrow v = \omega r_{pulley}$
- Solid disk $\rightarrow I = \frac{1}{2}M_{pulley}(r_{pulley})^2$
- Putting it together: $(v_f)^2 = (40/7)(m/s)^2 \rightarrow v_f = 2.39$ m/s
- Need acceleration to get the time; can assume acc. is constant, so $a = (v_f)^2 / 2h = 20/7$ m/s^2. $t = (2h/a)^{½} = 0.84$ s

To come:

- **Rigid body equilibrium;**
- **Cross product of vectors;**
- **Angular momentum;**

Assignment:

- **Continue reading and working on Chapter 12, especially the topics above.**