V Radiation Systems [Jackson chapt. 9, " 14]

→ Radiation by varying currents / charges.

→ Radiation as a ~~difficult~~ boundary value problem.

→ Multipoles expansion.

→ Radiation by moving charges.

1.) Radiation:

a.) in (x,t)-space : is induced by any variation of currents which because of the wave propagate.

In Vacuum, the vector potential $\vec{A}(\vec{r},t)$ in Lorentz-gauge is

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3\vec{r}\,'\int dt'\,\frac{\vec{j}(\vec{r}\,',t')}{|\vec{r}-\vec{r}\,'|}\,\delta\left[t'-\left(t-\frac{|\vec{r}-\vec{r}\,'|}{c}\right)\right]$$

$$\Longleftrightarrow \frac{1}{|\vec{r}-\vec{r}\,'|}\,\delta\left[t'-\left(t-\frac{|\vec{r}-\vec{r}\,'|}{c}\right)\right]$$ is the retarded Greens function of the wave equation in (x,t) space.

Note: We could also formulate it as a variation of charges (instead currents) - but they are related to those of the current through charge conversation $\left(\nabla\cdot\vec{j}+\frac{\partial\rho}{\partial t}=0\right)$ which is built in the Maxwell equation!

$\Longrightarrow$

b.) $(\vec{r},\omega)$ - space:

If we take the time Fourier transform - or consider a single frequency component

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} e^{-i\omega t} \int d^3\vec{r}' \, \vec{j}(\vec{r}') \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}$$

where $k \equiv \omega/c$

$$\rightsquigarrow \vec{A}(\vec{r},t) = \vec{A}(\vec{r}) \cdot e^{-i\omega t}$$

$$\text{with} \quad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3\vec{r}' \, \vec{j}(\vec{r}') \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}$$

Note: we have not taken a $\vec{r}$ Fourier transform

$\Longleftrightarrow$ $-\dfrac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}$ is the retarded Greens function of the wave equation in $(\vec{r},\omega)$ - space!

c.) Fields:

Outside of the sources, they can be obtained from:

$$\vec{H} = \frac{1}{\mu_0} \nabla \times \vec{A} \qquad Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \text{impedance of free space}$$

$$\varepsilon_0 \dot{\vec{E}} = -i\omega\varepsilon_0 \vec{E} = \nabla \times \vec{H}$$

$$\Rightarrow \vec{E} = \frac{i}{k} \cdot \sqrt{\frac{\mu_0}{\varepsilon_0}} \nabla \times \vec{H} = \frac{i \cdot Z_0}{k} \nabla \times \vec{H}$$

$\Longrightarrow$

2. Spatial Regions:

Exact expansion of the Green's function (see Jackson Chapt. 3)

↳ It can be shown that

$$\frac{e^{ik(\vec{r}-\vec{r}\,')}}{|\vec{r}-\vec{r}\,'|} = -ik \cdot \sum_{\ell=0}^{\infty} j_\ell(kr_<)\, h_\ell^{(1)}(kr_>) \sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\theta',\varphi')\, Y_{\ell m}(\theta,\varphi)$$

with $j_\ell(x) = \sqrt{\frac{\pi}{2x}} \cdot J_{\ell+1/2}(x)$ is the regular Bessel function

$$h_\ell^{(1)}(x) = \sqrt{\frac{\pi}{2x}} \cdot \left[J_{\ell+1/2}(x) + i\, N_{\ell+1/2}(x) \right]$$

$$= \sqrt{\frac{\pi}{2x}}\, H_{\ell+1/2}^{(1)} \quad \text{is the spherical Hankel function}$$

$r_<$ is the smallest of $|\vec{r}|$ and $|\vec{r}\,'|$

$r_>$ " " largest " "

a.) <u>Near zone:</u> $r' \ll r \ll \lambda$

"static"

$$\rightsquigarrow j_\ell(k \cdot r') \approx \frac{(kr')^\ell}{(2\ell+1)!} ;$$

[static in character: $e^{ik(\vec{r}-\vec{r}\,')} \approx 1$ just $e^{-i\omega t}$]

$$h_\ell^1(k \cdot r) \approx -\frac{i(2\ell-1)!}{(k \cdot r)^{\ell+1}}; \quad \left[(2\ell+1)! \equiv (2\ell+1)\cdot(2\ell-1)\ldots(3)\cdot(1) \right]$$

$$\hookrightarrow A(\vec{r},t) = \frac{\mu_0}{4\pi} \cdot e^{-i\omega t} \cdot \sum_{\ell m} \frac{4\pi}{(2\ell+1)} \cdot \frac{Y_{\ell m}(\theta,\varphi)}{r^{\ell+1}} \int d^3\vec{r}\,'\, \vec{J}(\vec{r}\,')\, r'^{\ell}\, Y_{\ell m}^*(\theta',\varphi') \Rightarrow$$

b.) Intermediate zone: $r' \ll r \approx \lambda$

"Induction"

↳ we will need full expression of Green's function!

c.) Far zone: $r' \ll \lambda \ll r$

"Radiation"

We can use $h_l^1(kr) \approx (-i)^{l+1} \cdot \frac{e^{ikr}}{r}$

$$j_l(k \cdot r') \approx \frac{r'^l}{(2l+1)!}$$

However, it is simpler to proceed by expanding

$$|\vec{r} - \vec{r}'| \approx r - \hat{r} \cdot \vec{r}'$$

↳ Expand the phase term

$$A(\vec{r}, t) \approx \frac{\mu_0}{4\pi} e^{-i\omega t} \cdot \frac{e^{ikr}}{r} \int d^3\vec{r}' \; \vec{J}(\vec{r}') \, e^{-ik \cdot (\hat{r} \cdot \vec{r}')}$$

$$\approx \frac{\mu_0}{4\pi} e^{-i\omega t} \frac{e^{ik \cdot r}}{r} \sum_n \frac{(-i \cdot k)^n}{n!} \underbrace{\int d^3\vec{r}' \; \vec{J}(\vec{r}') \cdot (\hat{r} \cdot \vec{r}')^n}_{\text{usually 2: electric + magnetic}}$$

↳ expression is dominated by first term which is not zero!

3.) Electric Dipole & Radiation

If only the first term is kept

$$\hookrightarrow A(\vec{r},t) \approx \frac{\mu_0}{4\pi} e^{-i\omega t} \frac{e^{ikr}}{r} \sum_n \frac{(-ik)^n}{n!} \int d^3r' \, \vec{J}(\vec{r}') \, (\hat{r}\cdot\vec{r}')^n$$

$$\llcorner n=0 \hookrightarrow \boxed{A(\vec{r},t) = e^{-i\omega t} \cdot \Big[\underbrace{\frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \cdot \int d^3r' \, \vec{J}(\vec{r}')}_{= A(\vec{r})} \Big]}$$

This is the "$l=0$" part of the series $\Rightarrow$ valid everywhere outside the source – not just in far-zone.

use continuity equ. $\quad i\omega\rho = \nabla\cdot\vec{J}$

to rewrite $\displaystyle \int \vec{J}(\vec{r}')\, d^3r' = -\int \vec{r}'(\nabla'\cdot\vec{J})\, d^3r' = -i\omega \int \vec{r}' \rho(\vec{r}')\, d^3r'$

$$\hookrightarrow \vec{A}(\vec{r}) = -\frac{i\omega\mu_0}{4\pi} \cdot \frac{e^{ikr}}{r} \cdot \underbrace{\int \vec{r}'\rho(\vec{r}')\, d^3r'}_{=:\vec{p}}$$

$=: \vec{p}$ – electric dipole defined in electrostatic

$$\hookrightarrow \boxed{\vec{A}(\vec{r},t) = -\frac{i\omega\mu_0}{4\pi} \cdot \vec{p} \cdot \frac{e^{ikr}}{r} \cdot e^{-i\omega t}}$$

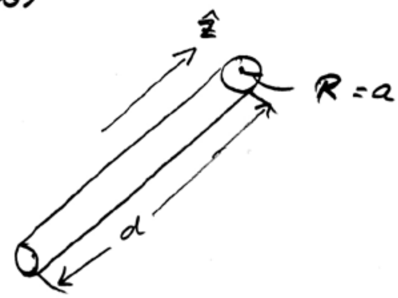
fields: $\displaystyle \vec{H} = \frac{1}{\mu_0}\left(\nabla\times\vec{A}\right) = \frac{ck^2}{4\pi}\left(\hat{r}\times\vec{p}\right)\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)$

$$E = \frac{iZ_0}{k}\left(\nabla\times\vec{H}\right) = \frac{1}{4\pi\varepsilon_0}\Big[k^2(\hat{r}\times\vec{p})\times\hat{r}\,\frac{e^{ikr}}{r} + \left(3\cdot\hat{r}(\hat{r}\cdot\vec{p}) - \vec{p}\right)\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)e^{ikr} \Big]$$

Time-average Power radiated per solid angle: $\displaystyle \frac{dP}{d\Omega} = \frac{1}{2}\,\mathrm{Re}\left[r^2 \cdot \hat{r}\cdot\vec{E}\times\vec{H}^*\right]$

The Antenna as a boundary value problem

Let's assume that we have a cylindrical antenna of finite length $\underline{d}$. If it has a radius a, and is a perfect conductor, one of the boundary condition is: $E_z(\rho=0) = 0$



In contrary to the waveguide case, the current over the cross-section is not constant $\rightsquigarrow$ no simple way to satisfy the boundary condition.

Using the Lorentz-gauge: $\frac{1}{c^2}\underbrace{\frac{\partial \phi}{\partial t}}_{\text{scalar-}} + \nabla \cdot \underbrace{\vec{A}}_{\text{vector pot.}} = 0$

$$\Updownarrow$$

$$\phi(\vec{r}) = -\frac{ic}{k} \cdot (\nabla \cdot \vec{A}) \quad \text{for frequency } \underline{\omega}$$

we get

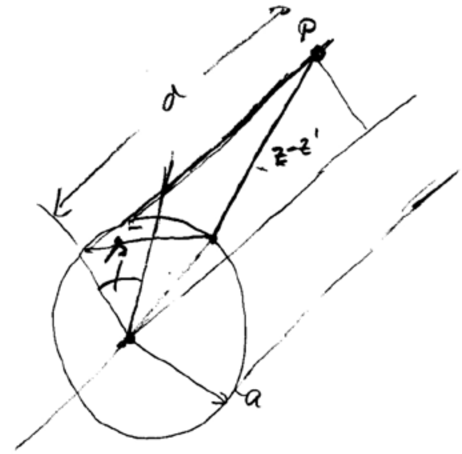
$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} = \frac{ic}{k}\left[\nabla(\nabla\cdot\vec{A}) + k^2 \vec{A}\right]$$

Since $\vec{A} = \hat{z} \cdot A_z(\vec{r}) \rightsquigarrow \boxed{E_z(\vec{r}) = \frac{ic}{k}\left[\frac{\partial^2}{\partial z^2} + k^2\right] A_z(\vec{r})}$

On the surface of the antenna: $E_z(\rho = a) = 0$

$$\hookrightarrow \left[\frac{\partial^2}{\partial z^2} + k^2\right] A_z(\rho = 0)$$

$\rightsquigarrow$ $A_z(\rho = a)$ is strictly <u>sinusoidal</u>! $\Rightarrow$



$\rightsquigarrow$ But: $A_z(\rho = a) = \frac{\mu_0}{4\pi} \int_{z'=0}^{z'=d} J(z')\, K(z-z')\, dz'$

with $\vec{J}(x,y,z) = J(z)\, \delta(\rho - a)$

where $K(z-z') = \frac{1}{\pi} \int_0^{\pi} \frac{e^{ik \cdot R}}{R}\, d\beta$

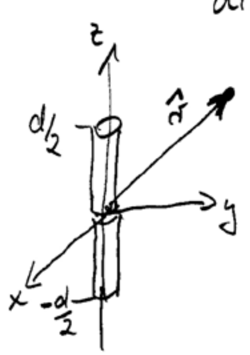
is the azimuthal average of $\frac{e^{ikR}}{R}$, $\beta = \varphi/2$

$\hookrightarrow K(z-z') = \frac{1}{\pi} \int_0^{\pi} \frac{e^{ik \cdot \sqrt{(z-z')^2 + 4a^2 \sin^2\beta}}}{\sqrt{(z-z')^2 + 4a^2 \sin^2\beta}} \cdot d\beta$

To solve A_z, we have to find $J(z')$ as a solution of the integro-differential equation

$$\boxed{\left[\frac{\partial^2}{\partial z^2} + k^2 \right] \int_{z'=0}^{z'=d} J(z') \cdot K(z-z')\, dz' = 0}$$

Example: Center fed, linear antenna with $d \ll \lambda$, oriented along z-axis with $z = [-d/2, d/2]$ and current is same direction in each half of antenna



$\rightsquigarrow J(z) \cdot e^{-i\omega t} = J_0 \left(1 - \frac{2|z|}{d} \right) e^{-i\omega t}$

use $i\omega\rho = \nabla \cdot \vec{J}$ $\rightsquigarrow \rho(z) = \pm \frac{2i J_0}{\omega d}$

$\vec{P} = \int_{-d/2}^{d/2} \hat{z}\, \rho(z)\, dz = \frac{i J_0 d}{2\omega} \hat{z}$ = dipole moment

5. Thin antenna approximation

If $a \ll \frac{1}{k} = \frac{2\pi}{\lambda}$

the term $K(z-z') = \frac{1}{\pi}\int_0^{\pi} \frac{e^{ik\sqrt{(z-z')^2+4a^2\cos^2\varphi}}}{\sqrt{(z-z')^2+4a^2\cos^2\varphi}}\, d\varphi$

will be very large when $z - z' < a$

i.e. $K(z-z') \approx f(a)\cdot\delta(z-z')$

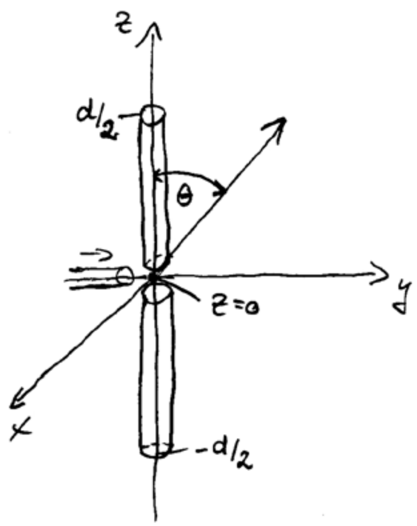
$$\hookrightarrow \int_{z'=0}^{z=d} J(z')\, K(z-z')\, dz' \approx J(z) \Rightarrow \boxed{\left[\frac{\partial^2}{\partial z^2} + k^2\right] J(z) = 0}$$

In this case, $J(z)$ is sinusoidal!

(Note: It does not converge very rapidly because of logarithms)

Example: Center fed linear antenna (Chapt. 9.4 Jackson p.416...)

Assume that the current is indeed sinusoidal



$$\vec{J}(\vec{r}) = J\cdot\sin\left(\frac{k\cdot d}{2} - k\cdot|z|\right)\delta(x)\,\delta(y)\cdot\hat{z}$$

for $|z| < d/2$

$\hookrightarrow$ In radiation zone:

$$A(\vec{r}) = \frac{\mu_0}{4\pi} J\cdot\hat{z}\,\frac{e^{ikr}}{r}\int_{-d/2}^{+d/2} \sin\left[\frac{k\cdot d}{2} - k\cdot|z|\right]\cdot e^{-ik\cdot\cos\theta}\, dz$$

$$= \frac{\mu_0}{2\pi} J\cdot\hat{z}\cdot\frac{e^{ikr}}{r}\cdot\left[\frac{\cos\left(\frac{kd}{2}\cdot\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right] \Rightarrow$$

$\rightsquigarrow$ In radiation zone:

$$\vec{H} = \frac{1}{\mu_0} \nabla \times \vec{A} \approx \frac{1}{\mu_0}\left(\hat{r}\frac{\partial}{\partial r}\right) \times \vec{A} \approx ik\,\hat{r} \times \vec{A}$$

(where we kept only $\frac{1}{r}$ - terms. $\nabla = \hat{r}\frac{\partial}{\partial r} + \theta \frac{\partial}{r\partial\theta} + \hat{\varphi}\frac{\partial}{r\sin\theta\partial\theta}$)

$$\vec{E} \approx -\sqrt{\frac{\mu_0}{\varepsilon_0}}\;\hat{r} \times \vec{H}$$

$$\hookrightarrow \frac{dP}{d\Omega} = \frac{1}{8\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}} \cdot J^2 \cdot \left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{k}{d}\right)}{\sin\theta}\right]^2$$

angular distribution depends on value k·d!

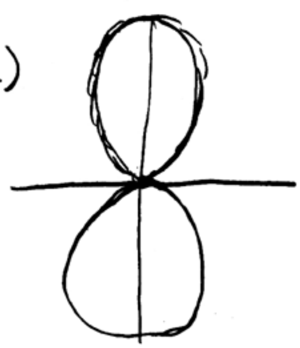
For $k \cdot d \ll 1$: $\frac{dP}{d\Omega} \approx \frac{Z_0 J^2}{128\pi^2} \cdot (kd)^2 \sin^2\theta$

For special values of kd :

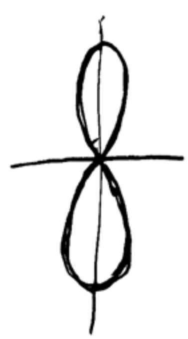
a.) $kd = \pi \rightsquigarrow \frac{dP}{d\Omega} = \frac{Z_0 J^2}{8\pi^2} \cdot \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$

b.) $kd = 2\pi \rightsquigarrow \frac{dP}{d\Omega} = \frac{4 Z_0 J^2}{8\pi^2} \frac{\cos^4\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$

Radiation pattern: a.)



b.)



6. Multipole Expansion

In empty space, the fields are completely determined by their components along the line of sight:

$$\vec{r}\cdot\vec{H} \quad \text{and} \quad \vec{r}\cdot\vec{E}$$

Introduce angular momentum operator: $\boxed{\vec{\mathcal{L}} = \frac{1}{i}(\vec{r}\times\nabla)}$

! $\vec{\mathcal{L}}$ acts only on angles and $\vec{r}\cdot\vec{\mathcal{L}} = 0$!

Remember from Quantum mechanics:

$$L^2 = L_x^2 + L_y^2 + L_z^2 ;$$

$$L_+ = L_x + iL_y = e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right);$$

$$L_- = L_x - iL_y = e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right);$$

$$L_z = -i\frac{\partial}{\partial\phi}$$

with operation on the spherical harmonics $Y_{\ell m}(\theta,\phi)$

$$L^2\, Y_{\ell m} = \ell\cdot(\ell+1)\, Y_{\ell m}$$

$$L_+\, Y_{\ell m} = \sqrt{(\ell-m)(\ell+m+1)}\; Y_{\ell,m+1}$$

$$L_-\, Y_{\ell m} = \sqrt{(\ell+m)(\ell-m+1)}\; Y_{\ell,m-1}$$

$$L_z\, Y_{\ell m} = m\, Y_{\ell m}$$

next, define the spherical vector harmonic $\vec{X}_{\ell m}(\theta,\phi)$

$$\boxed{\vec{X}_{\ell m}(\theta,\phi) = \frac{1}{\sqrt{\ell\cdot(\ell+1)}}\,\vec{\mathcal{L}}\,Y_{\ell m}(\theta,\phi)}$$

with
$$\left.\begin{aligned} &\int \vec{X}^{*}_{\ell' m'}\cdot\vec{X}_{\ell m}\,d\Omega = \delta_{\ell\ell'}\,\delta_{m m'} \\ &\int \vec{X}^{*}_{\ell' m'}\cdot(\vec{r}\times\vec{X}_{\ell m})\,d\Omega = 0 \end{aligned}\right\}\ \text{orthogonality properties}$$

Define $f_\ell(kr) = A_\ell^{(1)}\,h_\ell^{(1)}(kr) + A_\ell^{(2)}\,h_\ell^{(2)}(kr)$ =: radial functions

with $A_\ell^{(1)}, A_\ell^{(2)}$ are constants to be determined

$$h_\ell^{(1,2)}(x) = \sqrt{\frac{\pi}{2x}}\left(J_{\ell+1/2}(x) \pm i\,N_{\ell+1/2}(x)\right)$$ are the spherical Hankel functions; $h_\ell^{(2)}(x) = h_\ell^{(1)*}(x)$

↳ Magnetic (= transverse electric) Multipole : $\vec{r}\cdot\vec{E}^{M}_{\ell m} = 0$

$$\vec{E}^{M}_{\ell m} \propto f_\ell(kr)\cdot\vec{X}_{\ell m}$$

$$\vec{H}^{M}_{\ell m} = \frac{-i}{k\cdot Z_0}\,\nabla\times E^{M}_{\ell m} \qquad \left(Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}},\ k = \frac{\omega}{c}\right)$$

↳ Electric (= transverse magnetic) multipole : $\vec{r}\cdot H^{E}_{\ell m} = 0$

$$\vec{H}^{E}_{\ell m} \propto f_\ell(kr)\cdot\vec{X}_{\ell m}$$

$$E^{E}_{\ell m} = \frac{i Z_0}{k}\,\nabla\times\vec{H}^{E}_{\ell m}$$

Expansion of a general field in empty space

↳ take a second radial function $g_\ell(kr) = B_\ell\, h_\ell^{(1)}(kr) + B_\ell^{(2)}\, h_\ell^{(2)}(kr)$

$$\hookrightarrow \vec{H} = \sum_{\ell m}\left[a_{\ell m}^{E} f_\ell(kr)\, \vec{X}_{\ell m} - \frac{i}{k} a_{\ell m}^{M}\, \nabla \times g_\ell(kr)\, \vec{X}_{\ell m} \right]$$

$$\vec{E} = Z_0 \sum_{\ell m}\left[\frac{i}{k} a_{\ell m}^{E}\, \nabla \times f_\ell(kr)\, \vec{X}_{\ell m} + a_{\ell m}^{M} \cdot g_\ell(kr)\, \vec{X}_{\ell m} \right]$$

with: $$Z_0\, a_{\ell m}^{E} f_\ell(kr) = -\frac{k}{\sqrt{\ell\cdot(\ell+1)}} \int Y_{\ell m}^{*}\, \vec{r}\cdot\vec{E}\, d\Omega$$

$$a_{\ell m}^{M} g_\ell(kr) = \frac{k}{\sqrt{\ell\cdot(\ell+1)}} \cdot \int Y_{\ell m}^{*}\, \vec{r}\cdot\vec{H}\, d\Omega$$

To get a complete solution, integration needs to be done at two different radii r_1 and r_2!

Radiation zone

Consider outgoing waves

$$f_\ell, g_\ell \propto h_\ell^{(1)} \xrightarrow[r \to \infty]{} (-i)^{\ell+1} \frac{e^{ikr}}{k\cdot r}$$

↳ Fields: $$\vec{H}(\vec{r},t) = \frac{e^{i(kr-\omega t)}}{r} \cdot \sum_{\ell m} (-i)^{\ell+1} \left[a_{\ell m}^{E} \vec{X}_{\ell m} + a_{\ell m}^{M}\, \hat{r}\times\vec{X}_{\ell m} \right]$$

$$\vec{E}(\vec{r},t) = -\sqrt{\frac{\mu_0}{\varepsilon_0}}\, \hat{r}\times\vec{H}(\vec{r},t)$$

⇒

Angular distribution of radiation:

time-average power radiated per solid angle:

$$\frac{dP}{d\Omega} = \frac{1}{2k^2}\sqrt{\frac{\mu_0}{\varepsilon_0}} \cdot \Big| \sum_{l,m} (-i)^{l+1} \left[-a^E_{lm} \left(\hat{r} \times \vec{X}_{lm}\right) + a^M_{lm} \vec{X}_{lm} \right] \Big|^2$$

For one single multipole, the angular distribution reduces to a single term:

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2} \, |a_{lm}|^2 \cdot |\vec{X}_{lm}|^2$$

what is a combination of $|Y_{lm}|^2$, $|Y_{l(m+1)}|^2$, $|Y_{l(m-1)}|^2$

since $X_{lm}(\theta,\phi) = \frac{1}{\sqrt{l(l+1)}} \vec{\mathcal{L}} \cdot Y_{lm}(\theta,\phi)$

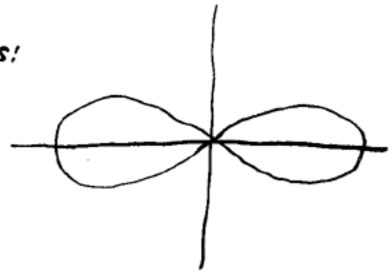
Applying the relations for L_+, L_-, and L_z on Y_{lm}

$$\hookrightarrow \frac{dP}{d\Omega} = \frac{Z_0 \cdot |a_{lm}|^2}{2k^2} \cdot \left\{ \frac{1}{2}(l-m)(l+m+1)\left[Y_{l,m+1}\right]^2 + \frac{1}{2}(l+m)(l-m+1)\left[Y_{l,m-1}\right]^2 + m^2\left[Y_{lm}\right]^2 \right\}$$

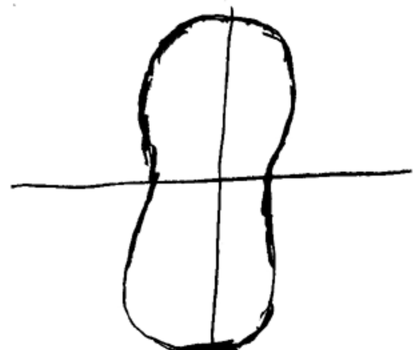
$|X_{lm}(\theta,\phi)|^2$:

l ↓ / m →	0	±1	±2
$l=1$ (Dipole)	$\frac{3}{8\pi}\sin^2\theta$	$\frac{3}{16\pi}(1+\cos^2\theta)$	—
$l=2$ Quadrupole	$\frac{15}{8\pi}\sin^2\theta\cos^2\theta$	$\frac{5}{16\pi}(1-3\cos^2\theta+4\cos^4\theta)$	$\frac{5}{16\pi}(1-\cos^4\theta)$

⤳ Radiation patterns:

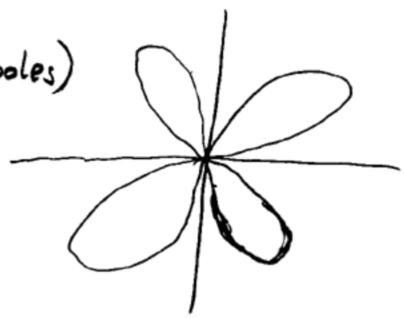


$\ell = 1,\ m = 0$

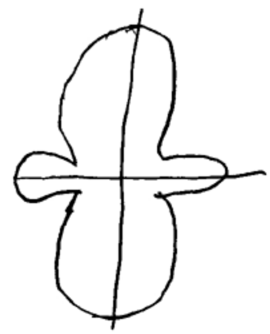


$\ell = 1,\ m = \pm 1$

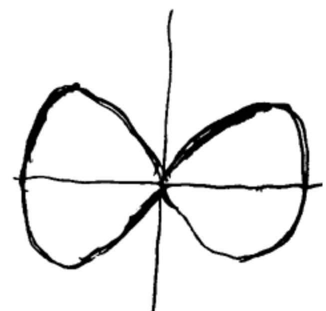
$\ell = 2$ (Quadrupoles)



$m = 0$



$m = \pm 1$



$m = \pm 2$

Total power radiated

$$\boxed{P = \frac{1}{2k^2}\sqrt{\frac{\mu_0}{\varepsilon_0}} \cdot \sum_{\ell m}\left[\,|a^E_{\ell m}|^2 + |a^M_{\ell m}|^2\,\right]}$$

= sum of contributions from different multipoles.