6. Lorentz invariant field theory

Read: Chapt. 12, Jackson pp. 579 - 615

6.1. Classical Hamiltonian for particles and fields:

In classical mechanics, the Hamiltonian variation principle for a particle is given by

$$\delta \int_{t_1}^{t_2} \mathcal{L}(x_k, \dot{x}_k, t)\, dt = 0 \qquad \Big|\ \dot{x}_k = \frac{d x_k}{dt}$$

with $\mathcal{L}$:= Lagrange function, which has for conservative forces the form:

$$\mathcal{L}(x_k, \dot{x}_k, t) = \underbrace{T(\dot{x}_k)}_{\substack{=\text{kinetic}\\ \text{energy}}} - \underbrace{V(x_k, t)}_{\substack{=\text{potential}\\ \text{energy}}}$$

A similar form can be found also for many non-conservative forces. An example is the movement of an elemental particle with charge $e = q$ in an electro-magnetic field:

$$m \cdot \ddot{\vec{r}} = e\,(\vec{E} + \dot{\vec{r}} \times \vec{B})$$

with the Lagrange function

$$\mathcal{L} = \frac{m}{2}\dot{\vec{r}}^{\,2} + e \cdot \dot{\vec{r}} \cdot \vec{A} - e \cdot \varphi = T - V$$

satisfying the Euler-Lagrange Equation:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = 0$$

$\rightarrow$

⇝ Also follows from the Lagrange function

$$\mathcal{L} = \frac{m\dot{\vec{r}}^2}{2} + e\dot{\vec{r}}\cdot\vec{A} - e\varphi$$

$$= -m_0c^2\sqrt{1-\frac{v^2}{c^2}} + e(\dot{\vec{r}}\cdot\vec{A} - \varphi) \quad \Big| \frac{mv^2}{2} \to m_0c^2\sqrt{1-\frac{v^2}{c^2}}$$

the relativistic equation of motion:

$$\frac{d}{dt}\vec{p} = \frac{d}{dt}\left[\frac{m_0\dot{\vec{r}}}{\sqrt{1-v^2/c^2}}\right] = e(\vec{E} + \dot{\vec{r}}\times\vec{B})$$

Using the Lagrange function and the generalized momentum $p_k := \frac{\partial \mathcal{L}}{\partial \dot{q}_k}$, we can define the Hamiltonian

$$H(p_k, q_k, t) = \sum_k p_k \cdot \dot{q}_k - \mathcal{L}$$

For the case of conservative forces:

$$H = T - V$$

For a charged particle in an electromagnetic field, we obtain

$$H = \vec{p}\cdot\dot{\vec{r}} - \mathcal{L} = \frac{1}{2}m\cdot\dot{\vec{r}}^2 + e\cdot\varphi - e\cdot\dot{\vec{r}}\cdot\vec{A}$$

→

→ Similarly follows for a relativistic particle:

$$\vec{p} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} = \frac{m_0 \dot{\vec{r}}}{\sqrt{1 - \dot{\vec{r}}^2/c^2}} + e \cdot \vec{A}$$

$$\hookrightarrow H = \vec{p} \cdot \dot{\vec{r}} - \mathcal{L} = \frac{m_0 c^2}{\sqrt{1 - \dot{\vec{r}}^2/c^2}} + e \cdot \varphi$$

If we now move from particles over to fields, the generalized Hamiltonian variation principle becomes:

$$\delta \int_{t_0}^{t_1} dt \int d^3\vec{r} \; \mathcal{L}(\varphi_k, \varphi_{k/l}, \dot{\varphi}_k, t) = 0$$

$$\left[\text{with } \varphi_{k/l} := \frac{\partial \varphi_k}{\partial x_l}, \quad \dot{\varphi}_k = \frac{\partial \varphi_k}{\partial t}\right]$$

Satisfying the Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \varphi_\nu} - \sum_{k=1}^{3} \frac{\partial}{\partial x_k}\left(\frac{\partial \mathcal{L}}{\partial \varphi_{\nu/k}}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}_\nu}\right) = 0$$

For electromagnetic fields with $\rho = 0$, $\vec{j} = 0$ follows from the Lagrange density

$$\mathcal{L} = \frac{1}{2} \varepsilon_0 \dot{\vec{A}}^2 - \frac{1}{2} \cdot \frac{1}{\mu_0} (\nabla \times \vec{A})^2 \qquad (\text{with } \nabla \vec{A} = 0)$$

the field equation

$$\boxed{\Box \vec{A} = \left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = 0}$$

$\Rightarrow$

$\rightsquigarrow$ The Euler-Lagrange equations for the fields are easily formulated in 4D-vector description:

$$\boxed{\frac{\partial \mathcal{L}}{\partial \psi_\nu} - \partial_\varrho \frac{\partial \mathcal{L}}{\partial_\varrho \psi_\nu} = 0} \qquad \text{(Euler-Lagrange)}$$

With $\mathcal{L} = -\frac{1}{4} F_{\nu\mu} F^{\nu\mu} + \mu_0 A_\mu S^\mu \left(= \frac{1}{2}\varepsilon_0 \dot{\vec{A}}^2 - \frac{1}{2\mu_0} (\nabla \times \vec{A})^2 \right)$

we can reconstruct our Maxwell equations!

<u>set:</u> $F_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu \cdot A_\nu$

than the homogeneous Maxwell's equations are satisfied automatically. The Euler-Lagrange Equ. contain than the inhomogeneous DE's:

1.) $\mathcal{L} = -\frac{1}{4} (\partial_\nu A_\mu - \partial_\mu A_\nu)(\partial^\nu A^\mu - \partial^\mu A^\nu) + \mu_0 A_\mu S^\mu$

$$= -\frac{1}{2} \left[\partial_\nu A_\mu \partial^\nu A^\mu - \partial_\nu A_\mu \partial^\mu A^\mu \right] + \mu_0 A_\mu S^\mu$$

$$\hookrightarrow \frac{\partial \mathcal{L}}{\partial A_\nu} = \mu_0 S^\mu$$

2.) $\partial_\varrho \frac{\partial \mathcal{L}}{\partial \partial_\varrho A_\nu} = - \partial_\varrho \partial^\varrho A^\nu - \partial_\varrho \partial^\nu A^\varrho = \partial_\varrho F^{\nu\varrho}$

$$\hookrightarrow \boxed{\partial_\varrho F^{\nu\varrho} = \mu_0 S^\nu}$$

inhomog. field equation
(see p. 13/14)
$\hookrightarrow$ Maxwell Equations //

6.2 Lorentz invariant Hamiltonian principle (Chapt. 12 Jackson)

In order to formulate a Lorentz invariant theory for the motion of charged particles in an electromagnetic field, we have to formulate the Hamiltonian principle as Lorentz-invariant.

In the rest-system of a mass point, we have as the invariance:

$$\boxed{ds^2 = dx_\nu dx^\nu = -c^2 d\tau^2} \quad (\tau := \text{rest-time})$$

We can formulate a Lorentz invariant (inter)-action integral for a free particle by

$$W_{FT} = i \int m_0 c \, \underbrace{ds}_{-ic\cdot d\tau} \qquad (FT: \text{free particle})$$

From this follows

$$W_{FT} = i \int m_0 c \; ds = -m_0 c^2 \int_{\tau_1}^{\tau_2} d\tau = -m_0 c^2 \int_{t_1}^{t_2} \frac{1}{\gamma} dt$$

$$= -\int_{t_1}^{t_2} m_0 \cdot c^2 \sqrt{1 - \frac{\dot{\vec{r}}^2}{c^2}} \; dt$$

$$= -\int_{t_1}^{t_2} \underbrace{\mathcal{L}_{FT}}_{\text{= Lagrange function for a free particle}} dt$$

(see Chapt. 6.1 / p. 20)

$\Rightarrow$

~> Next, we express the action integral in the 4D-vector description, considering that

$$E_{FT} = H_{FT} = \text{Hamilton function}$$

and $L_{FT} = \vec{p} \cdot \dot{\vec{r}} - H_{FT}$

↳ from which follows

$$W_{FT} = \int_{t_1}^{t_2} L_{FT}\, dt = \int_{t_1}^{t_2} (\vec{p} \cdot \dot{\vec{r}} - H_{FT})\, dt$$

$$= \int_A^B (\vec{p} \cdot d\vec{r} - E_{FT} \cdot dt) = \int_A^B p_\nu \, dx^\nu$$

↳ $\boxed{W_{FT} = \int_A^B p_\nu \, dx^\nu}$ $\left(= \int_{t_1}^{t_2} L_{FT}\, dt \right)$

with $p_\nu = (p_1, p_2, p_3, -E/c)$

If we move now a particle in a EM-field, it deemed to be:

$$L = L_{FT} + e \cdot (\vec{A} \cdot \dot{\vec{r}} - \varphi)$$

If we set now

$$W = W_{FT} + \frac{1}{c} \int A_\nu \, j^\nu d^4x$$

than the additional term $\frac{1}{c} \int A_\nu j^\nu d^4x$ is Lorentz invariant since $d^4x = \det(L_\mu^\nu) d^4x_0 = d^4x_0$, resulting in the correct Lagrange function!

=>

→ To prove it, consider:

$$\frac{1}{c}\int A_\nu j^\nu d^4x = \int \vec{A}\cdot\vec{j}\, d^3\vec{r}\, dt - \int \varphi\cdot\rho\, d^3\vec{r}\, dt$$

$$= \int_{t_1}^{t_2} \vec{A}\cdot e\dot{\vec{r}}\, dt - \int_{t_1}^{t_2} e\,\varphi\, dt$$

using the fact that for a charged particle:

$$\vec{j}(\vec{r}\,') = e\cdot\vec{r}\,\delta(\vec{r}\,'-\vec{r})$$

$$\rho(\vec{r}\,') = e\cdot\delta(\vec{r}\,'-\vec{r})$$

From this follows:

$$\int A(\vec{r}\,')\, j(\vec{r}\,')\, d^3\vec{r}\,' = \int e\cdot\dot{\vec{r}}\; A(\vec{r}\,')\,\delta(\vec{r}\,'-\vec{r})\, d^3\vec{r}\,' = e\cdot\dot{\vec{r}}\cdot\vec{A}(\vec{r}(t))$$

and

$$\int \varphi(\vec{r}\,')\,\rho(\vec{r}\,')\, d^3\vec{r}\,' = \int \varphi(\vec{r}\,')\; e\cdot\delta(\vec{r}\,'-\vec{r})\, d^3\vec{r}\,' = e\cdot\varphi(\vec{r}(t))$$

The form invariant action function for the Lorentz transformation, which generates the correct motion equation is therefore – together with the associated Hamiltonian principle:

$$\boxed{W = \underbrace{\int_A^B p_\nu\, dx^\nu}_{\text{kinetic energy}} + \underbrace{\frac{1}{c}\int A_\nu j^\nu d^4x}_{\text{potential energy}}}$$

with $\boxed{\delta W = 0}$

6.3 Maxwell - Lorentz Theory

To summarize the theory of electromagnetic fields and charged particles in EM-fields, we will use an action integral consisting of three parts:

Part I: describe the free particle
" II: " " EM field
" III: " interaction between free particle and EM field!

To find such a action integral, it is sufficient to add to the action integral of a charged particle (last chapt. 6.2) that of the electromagnetic field:

$$W_{EM} = \frac{1}{4\mu_0 c}\int_A^B F_{\nu\mu}\, F^{\nu\mu}\, d^4x$$

(W_{EM} is Lorentz - invariant! → prove it!)

The generalized action function is therefore:

$$W = \underbrace{\int_A^B p_\nu\, dx^\nu}_{\text{free particle}} \underbrace{+\frac{1}{c}\int_A^B A_\nu\, j^\nu d^4x}_{\text{interaction particle} \leftrightarrow \text{EM field}} \underbrace{- \frac{1}{4\mu_0 c}\int_A^B F_{\nu\mu}\, F^{\nu\mu} d^4x}_{\text{EM-field}}$$

⟹

→ For the motion of a charged particle in the EM-field the action function has to be extremal:

$$\delta W = 0$$

This extremal principle can be translated in the following non-Lorentz invariant form:

$$\delta W = 0 \qquad \text{with} \quad W = \int_{t_1}^{t_2} \mathcal{L}\, dt$$

where

$$\mathcal{L} = \underbrace{-m_0 c^2 \sqrt{1-\frac{\dot{\vec{r}}^2}{c^2}}}_{\text{free particle}} + \underbrace{e\left(\vec{A}\cdot\dot{\vec{r}} - \varphi\right)}_{\substack{\text{interaction} \\ \text{particle} \leftrightarrow \text{EM field}}} + \underbrace{\frac{1}{2}\int\left(\vec{E}\cdot\vec{D} - \vec{H}\cdot\vec{B}\right) d^3\vec{r}\, dt}_{\text{EM-field}}$$

Since

$$F_{\nu\mu} \cdot F^{\nu\mu} = 2\cdot\left[B_x^2 + B_y^2 + B_z^2 - \frac{1}{c^2}\left(E_x^2 + E_y^2 + E_z^2\right)\right]$$

$$= 2\mu_0\left(\vec{E}\cdot\vec{D} - \vec{H}\cdot\vec{B}\right)$$

and

$$d^4x = c \cdot d^3\vec{r}\, dt$$

//

6.4. Lorentz invariant field equations

a.) The free EM-field:

The free EM-field is characterized by $\vec{j} = 0$ and $\rho = 0$ ($\rightsquigarrow s^\nu = 0$) and it describes phonons!

Maxwell - Equations: $\partial^\nu \partial_\nu A_\mu = 0 \quad \Leftrightarrow \quad \Box A_\nu = 0$

Lorentz - convention: $\partial^\nu A_\nu = 0$

Lagrange - density : $\mathcal{L} = \frac{-1}{4\mu_0 c} F_{\nu\mu} F^{\nu\mu}$

$$\hookrightarrow \mathcal{L} = \frac{-1}{4\mu_0 c} (\partial_\nu A_\mu - \partial_\mu A_\nu)(\partial^\nu A^\mu - \partial^\mu A^\nu)$$

$$= -\frac{1}{2\mu_0 c} \Big(\partial_\nu A_\mu \partial^\nu A^\mu - \underbrace{\partial_\nu A_\mu \partial^\mu A^\nu}_{= L_z} \Big)$$

Since $\partial_\rho A^\rho = 0$, the Euler-Lagrange equations vanish for $\mathcal{L}_z$ identical:

$$\frac{\partial \mathcal{L}_z}{\partial A_\nu} = 0 \; , \quad \partial_\rho \frac{\partial L_z}{\partial \partial_\rho A_\nu} = 2 \cdot \partial_\rho \partial^\nu A^\rho = 0$$

which reduces the Lagrange density to:

$$\boxed{\mathcal{L}' = -\frac{1}{2\mu_0 c} \partial_\nu A_\mu \partial^\nu A^\mu}$$

$\Rightarrow$

The Euler-Lagrange equation reproduce again:

$$\frac{\partial \mathcal{L}'}{\partial A_\nu} = 0 \quad , \quad \partial_\varsigma \frac{\partial \mathcal{L}'}{\partial \partial_\varsigma A_\nu} = -\frac{1}{\mu_0 c} \partial_\varsigma \partial^\varsigma A_\nu$$

$$\hookrightarrow \partial_\varsigma \partial^\varsigma A^\nu = 0 \quad \text{or} \quad \Box A^\nu = 0$$

According to the quantum mechanic theory, such a massless particle is described with spin 1. (= photon).

b.) <u>Free real scalar mesons</u>

This is according to quantum theory the simplest field that includes a particle with mass:

$$\varphi(x) : \text{real scalar} \quad , \quad x = (\vec{r}, ct)$$
$$\left[= A_\mu = (A_x, A_y, A_z, -\tfrac{1}{c}\varphi) \right]$$

For the Lagrange density we obtain

$$\mathcal{L} = -\frac{1}{2}\left[\partial^\nu \varphi \, \partial_\nu \varphi + m^2 \varphi^2 \right]$$

From the Euler-Lagrange equations follows therefore

$$\frac{\partial \mathcal{L}}{\partial \varphi} = -m^2 \varphi \; ; \; \partial_\varsigma \frac{\partial \mathcal{L}}{\partial \partial_\varsigma \varphi} = -\partial_\varsigma \partial^\varsigma \varphi \implies (\Box - m^2)\varphi = 0$$

(Klein-Gordon-Equa.)

After Quantum field theory: $\Rightarrow$ spinless particle with rest mass <u>m</u>. $\Rightarrow$

c) Free, real vector mesons

Free, real vector mesons of mass $\underline{m}$ are described by a vector field $\varphi_\nu(x)$, with the condition:

$$\partial^\nu \varphi_\nu(x) = 0$$

For the Lagrange density follows:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\nu \varphi_\mu \partial^\nu \varphi^\mu + m^2 \varphi_\nu \varphi^\nu\right)$$

Applying Euler-Lagrange equations, we obtain

$$\partial_\rho \partial^\rho \varphi^\nu = m^2 \varphi^\nu$$

or $$\Box \varphi^\nu = m^2 \varphi^\nu$$

After the quantum field theory, such a particle is described with rest mass m and spin 1.

↳ <u>problems</u> in special Relativity ⟹