4.) Electromagnetic field tensor

From the electromagnetic potentials $\vec{A}$ and φ (or A_μ) we can derivate the fields $\vec{E}$ and $\vec{B}$ according to

$$\vec{B} = \nabla \times \vec{A} \quad (\vec{A}: \text{Vector potential})$$

and
$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla\varphi = c \cdot \left[\nabla A_4 - \frac{\partial}{\partial x^4}\vec{A}\right]$$

$\vec{A}$ = 4D potential with $A_4 = -\frac{1}{c}\varphi$, $\frac{\partial}{\partial x^4} = -\frac{1}{c}\frac{\partial}{\partial t}$

For the single components we get:

$$B_x = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \qquad E_x = c \cdot \left(\frac{\partial A_4}{\partial x^1} - \frac{\partial A_1}{\partial x^4}\right)$$

$$B_y = \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \qquad E_y = c \cdot \left(\frac{\partial A_4}{\partial x^2} - \frac{\partial A_2}{\partial x^4}\right)$$

$$B_z = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \qquad E_z = c \cdot \left(\frac{\partial A_4}{\partial x^3} - \frac{\partial A_3}{\partial x^4}\right)$$

Let's next define the antisymmetric field tensor

$$F_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu$$

which has the form:

$$F_{\nu\mu} = \begin{pmatrix} 0 & B_z & -B_y & \frac{1}{c}E_x \\ -B_z & 0 & B_x & \frac{1}{c}E_y \\ B_y & -B_x & 0 & \frac{1}{c}E_z \\ -\frac{1}{c}E_x & -\frac{1}{c}E_y & -\frac{1}{c}E_z & 0 \end{pmatrix} \Rightarrow$$

→ we can group the four Maxwell equations

$$\boxed{\begin{array}{ll} \nabla\times E + \frac{\partial B}{\partial t} = 0 \quad ; & \nabla\times\vec{B} - \frac{1}{c^2}\frac{\partial E}{\partial t} = \mu_0\vec{j} \\ \nabla\cdot B = 0 \quad ; & \nabla\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho \end{array}}$$

in <u>two</u> differential equations (DE):

(*) $$\boxed{\partial_\nu F_{\mu\lambda} + \partial_\mu F_{\lambda\nu} + \partial_\lambda F_{\nu\mu} = 0}$$ "homogeneous field equation"

and

(**) $$\boxed{\partial^\nu F_{\nu\mu} = \mu_0 S_\nu}$$ "inhomogeneous field equation"

where (ν,μ,λ) is a cyclic permutation of $(1,2,3,4)$.
[e.g. $(\nu,\mu,\lambda) = (1,2,3)$ or $(2,3,4)$ or $(3,4,1)$ or $(4,1,2)\ldots$]

The homogeneous DE (*) contains four DE:

1.) $(\nu,\mu,\lambda) = (1,2,3)$

$$\hookrightarrow \frac{\partial F_{23}}{\partial x^1} + \frac{\partial F_{31}}{\partial x^2} + \frac{\partial F_{12}}{\partial x^3} = \frac{\partial B_x}{\partial x^1} + \frac{\partial B_y}{\partial x^2} + \frac{\partial B_z}{\partial x^3} = \nabla\cdot\vec{B} = 0$$

2.) $(\nu,\mu,\lambda) = (2,3,4)$

$$\frac{\partial F_{34}}{\partial x^2} + \frac{\partial F_{42}}{\partial x^3} + \frac{\partial F_{23}}{\partial x^4} = \frac{1}{c}\frac{\partial E_z}{\partial y} - \frac{1}{c}\frac{\partial E_y}{\partial z} + \frac{1}{c}\frac{\partial B_x}{\partial t} = 0$$

$$\Rightarrow \frac{1}{c}\left(\nabla\times\vec{E} + \frac{\partial B}{\partial t}\right)_x = 0 \qquad \Rightarrow$$

The other two DE contain the y- and z-components of the Faraday law.

The four inhomogeneous DE (**) contain the Ampere-Maxwell and Poisson equations:

$\nu = 1$:

$$\partial^\nu F_{1\mu} = \frac{\partial F_{12}}{\partial x_2} + \frac{\partial F_{13}}{\partial x_3} + \frac{\partial F_{14}}{\partial x_4}$$

$$= \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} - \frac{1}{c^2}\frac{\partial E_x}{\partial t} = \mu_0 j_x$$

...

$$\hookrightarrow \quad \boxed{\nabla \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j}}$$

$\nu = 4$:

$$\partial^\nu F_{4\mu} = \frac{\partial F_{41}}{\partial x_1} + \frac{\partial F_{42}}{\partial x_2} + \frac{\partial F_{43}}{\partial x_3}$$

$$= \frac{-1}{c}\frac{\partial E_x}{\partial x} - \frac{1}{c}\frac{\partial E_y}{\partial y} - \frac{1}{c}\frac{\partial E_z}{\partial z} = -\mu_0 c \cdot \rho$$

$$\hookrightarrow \quad \boxed{\nabla \cdot \vec{E} = \mu_0 c^2 \rho = \frac{1}{\varepsilon_0}\rho}$$ Poisson Equation

So far we have shown that the <u>two</u> DE (*) and (**) with the field tensor $F_{\nu\mu}$ reproduce the Maxwell equations in vacuum.

We still have to show the form invariance of the Maxwell equations $\Rightarrow$

against a Lorentz transformation!

For this, we have to prove that the field tensor $F_{\nu\mu}$ has the proper transformation properties, which we will show next! If we assume for right now that this is the case, it is easy to see that the inhomogeneous DE: $\partial^\nu F_{\nu\mu} = \mu_0 S_\nu$ has the correct transformation property, since ∂^ν and S_ν transform like contra- and co-variant vectors, respectively.

To prove that the homogeneous DE (*) has the proper transformation behavior, we introduce the anti-symmetric ε-tensor of 4^{th} degree:

$$\varepsilon_{ijkl} = \begin{cases} 1 & \text{for } (i,j,k,l) = \text{even permutations of } (1,2,3,4) \\ -1 & \text{for } (i,j,k,l) = \text{odd permutations of } (1,2,3,4) \\ 0 & \text{otherwise} \end{cases}$$

The homogeneous field equation (*) becomes with this:

$$\boxed{\varepsilon^{i\nu\mu\lambda} \cdot \partial_\nu F_{\mu\lambda} = 0}$$

This equation consists of 4 equations for $i = 1,2,3,4$. For $i=1$, we get:

$$\varepsilon^{1\nu\mu\lambda} \cdot \partial_\nu F_{\mu\lambda} = \partial_2 F_{34} - \partial_2 F_{43} + \partial_3 F_{42} - \partial_3 F_{24} + \partial_4 F_{23} - \partial_4 F_{32}$$

$$= 2[\partial_2 F_{34} + \partial_3 F_{42} + \partial_4 F_{23}] = 0! \quad \Longrightarrow$$

Since we pre-assumed that $F_{\nu\mu}$ has the correct transformation behavior, we have to show this now only for the ε-tensor.

$\rightsquigarrow$ We have to prove that

$$\varepsilon^{\nu\mu\rho\sigma} = L^{\nu}_{i}\, L^{\mu}_{j}\, L^{\rho}_{k}\, L^{\sigma}_{l}\, \varepsilon_0^{ijkl} \qquad (***)$$

has the same property as ε_0^{ijkl}.

The equation (***) can be written with help of determinants in the form of

$$\varepsilon^{\nu\mu\rho\sigma} = \begin{vmatrix} L^{\nu}_{1} & L^{\mu}_{1} & L^{\rho}_{1} & L^{\sigma}_{1} \\ L^{\nu}_{2} & L^{\mu}_{2} & L^{\rho}_{2} & L^{\sigma}_{2} \\ L^{\nu}_{3} & L^{\mu}_{3} & L^{\rho}_{3} & L^{\sigma}_{3} \\ L^{\nu}_{4} & L^{\mu}_{4} & L^{\rho}_{4} & L^{\sigma}_{4} \end{vmatrix}$$

Since for each L^{i}_{j} : $\det(L^{i}_{j}) = 1$, follow from the mathematical rule for determinats, that $\varepsilon^{\nu\mu\rho\sigma}$ also is a ε-tensor!

5.) Transformation properties of Field tensor

The field tensor $F_{\nu\mu}$ has the correct transformation behavior, if:

$$F_{\nu\mu} = L_{\nu}^{\ \rho}\, F_{0\rho\sigma}\, L_{\mu}^{\ \sigma}$$

$$\text{or} \qquad F_{0\alpha\beta} = L^{\nu}_{\alpha}\, F_{\nu\mu}\, L^{\mu}_{\beta} \qquad \Longrightarrow$$

$$\leadsto \quad F_{o\alpha\beta} = \begin{pmatrix} \gamma & 0 & 0 & \frac{v}{c}\cdot\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{v}{c}\cdot\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 & B_z & -B_y & \frac{1}{c}E_x \\ -B_z & 0 & B_x & \frac{1}{c}E_y \\ B_y & -B_x & 0 & \frac{1}{c}E_z \\ -\frac{1}{c}E_x & -\frac{1}{c}E_y & -\frac{1}{c}E_z & 0 \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & \frac{v}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{v}{c}\gamma & 0 & 0 & \gamma \end{pmatrix}$$

$$\hookrightarrow F_{\alpha\beta} = \begin{pmatrix} 0 & \gamma B_z - \frac{v}{c^2}\gamma E_y & -\gamma B_y - \frac{v}{c^2}\gamma E_z & \frac{1}{c}E_x \\ -\gamma B_z + \frac{v}{c^2}\gamma E_y & 0 & B_x & -\frac{v}{c}\gamma B_z + \frac{1}{c}\gamma E_y \\ \gamma B_y + \frac{v}{c^2}\gamma E_z & -B_x & 0 & \frac{v}{c}\gamma B_y + \frac{1}{c}\gamma E_z \\ -\frac{1}{c}E_x & \frac{v}{c}\gamma B_z - \frac{1}{c}\gamma E_y & -\frac{v}{c}\gamma B_y - \frac{1}{c}\gamma E_z & 0 \end{pmatrix}$$

On the otherside we defined the antisymmetric field tensor through

$$F_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu \rightsquigarrow F_{o\alpha\beta} = \begin{pmatrix} 0 & B_{oz} & -B_{oy} & \frac{1}{c}E_{ox} \\ -B_{oz} & 0 & B_{ox} & \frac{1}{c}E_{oy} \\ B_{oy} & -B_{ox} & 0 & \frac{1}{c}E_{oz} \\ -\frac{1}{c}E_{ox} & -\frac{1}{c}E_{oy} & -\frac{1}{c}E_{oz} & 0 \end{pmatrix}$$

By comparison of the components follows the tranformation behavior of the fields $\vec{E}$ and $\vec{B}$ from the 'rest'-frame in the 'lab'-frame:

$E_{ox} = E_x$	$B_{ox} = B_x$
$E_{oy} = \gamma(E_y - v\cdot B_z)$	$B_{oy} = \gamma(B_y - \frac{v}{c^2}\cdot E_z)$
$E_{oz} = \gamma(E_z + v B_y)$	$B_{oz} = \gamma(B_z - \frac{v}{c^2}E_y)$
$\hookrightarrow \vec{E}_o = \gamma[\vec{E} + v\times\vec{B}]$, $\vec{v} = (v,0,0)$	

$\Rrightarrow$

Now, we have to verify experimentally, that this is the correct transformation behavior.

First lets look at velocities small compared to the speed of light c:

$$\frac{v}{c} \ll 1 \quad \text{and} \quad \gamma \approx 1 \quad \text{and} \quad \vec{v} = \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \vec{E}_0 = \vec{E} + \vec{v} \times \vec{B}, \qquad \vec{B}_0 = \vec{B}$$

with the Coulomb-law in the 'rest'-frame follows

$$F_0 = q \cdot E_0 \qquad \text{'rest'-frame}$$

$$\Rightarrow \quad \vec{F} = q \cdot (\vec{E} + \vec{v} \times \vec{B}) \quad \text{in 'lab'-frame}$$

$\Rightarrow$ In the special relativity theory, the Lorentz force result from the transformation behavior of the field tensor $F_{\mu\nu}$, using the Coulomb law in the 'rest'-frame!

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