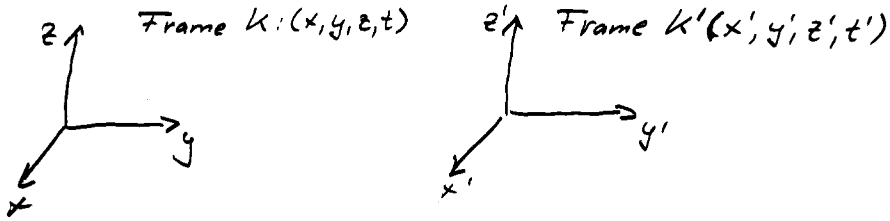
Special Theory of Relativity

0) Einstein's 2-postulates:

a.) assume 2 Frames:



↳ Galilean relativity: $x' = x - v \cdot t$, $t = t'$ ↯

Postulate: whatever given frame of reference, the results of all experiments have to be independent from frames moving with constant velocities to each other.

↳ combine space and time to 4-dim. Minkowski-space

b.) 2[nd] Postulate:

The speed of light **is**
- finite
- independent of the motion of the frame
- in every inertial frame, there is a finite universal limiting speed c for physical entities.

⟹

1.) Lorentz - Transformation

The combination of space and time to the 4-dim Minkowski-space leads to the expression $\boxed{x^2+y^2+z^2-c^2 \cdot t^2}$ which is invariant for Lorentz transformations.

Lets introduce a co- and contra-variant vector according to:

$$x^\nu = (x^1, x^2, x^3, x^4) = (x, y, z, c \cdot t) \quad | \text{ contra-variant}$$

$$x_\nu = (x_1, x_2, x_3, x_4) = (x, y, z, -c \cdot t) \quad | \text{ co-variant}$$

Then we have:

$$x^\nu \cdot x_\nu = \sum_{\nu=1}^{4} x^\nu \cdot x_\nu = x^2 + y^2 + z^2 - c^2 \cdot t^2$$

which is invariant against a Lorentz transformation.

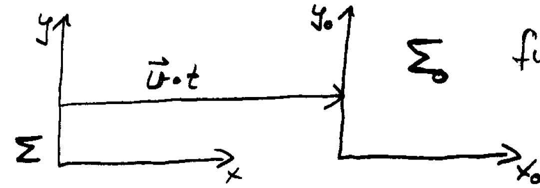
The Co- and Contra-variant vectors are connected over a metric fundamental tensor and can therefore be transformed in each other:

$$x_\nu = g_{\nu\mu} \cdot x^\mu \quad \text{and} \quad x^\mu = g^{\mu\nu} \cdot x_\nu$$

with

$$g_{\nu\mu} = g^{\nu\mu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Let's denote Σ as the Lab frame and Σ_0 as the origin frame of mass point:



Σ_0 fixed ~~or~~ static frame

then the contra-variant 4-vectors transform according to the Lorentz transformation:

$$x^\nu = L^\nu_\mu \, x_0^\mu$$

$$\text{with} \quad L^\nu_\mu(v) = \begin{pmatrix} \gamma & 0 & 0 & \gamma \cdot \frac{v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma \cdot \frac{v}{c} & 0 & 0 & \gamma \end{pmatrix} \; ; \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

↳ All vectors A^ν in the Minkowski space transform according to $\boxed{A^\nu = L^\nu_\mu \, A_0^{\ \mu}}$

This transformation behavior is not only valid for mechanical values e.g. space $\vec{r}$, velocity $\vec{v}$, momentum $\vec{p}$ or force $\vec{F}$ but also for the electrodynamic field vectors

⤳ Special Relativity Theory

Similarly to the introduction of the 4-D vectors, we can also formulate tensors for the Minkowski-space.

The requirement on such a tensor is that it behaves for each index the same way as ~~the~~ vectors: e.g. $T^\nu_\mu = g^{\nu\varrho} T_{\varrho\mu}$ ⇒

$\rightsquigarrow$ Transformation behavior for vectors & Tensors

i.) contra-variant vectors: $\boxed{x^{\nu} = L^{\nu}_{\mu}\, x_o^{\mu}}$

$$x_o^{\rho} = L_{\nu}^{\ \rho} L^{\nu}_{\mu}\, x_o^{\mu} = L^{\rho}_{\mu}\, x_o^{\mu}$$

since $L_{\nu}^{\ \rho} L^{\nu}_{\ \mu} = \delta^{\rho}_{\mu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\Rightarrow \quad \underset{\substack{\uparrow \\ \text{static-frame} \\ \text{vector}}}{x_o^{\rho}} = L_{\nu}^{\ \rho} \cdot \underbrace{x^{\nu}}_{\substack{\text{lab frame} \\ \text{vector}}} \iff \underset{\substack{\uparrow \\ \text{lab}}}{x^{\rho}} = L^{\rho}_{\ \nu}\, \underset{\substack{\uparrow \\ \text{static}}}{x_o^{\nu}}$$

The tensor $L_{\nu}^{\ \rho}$ satisfies the principles of relativity, since we can show that

$$L_{\nu}^{\ \rho} = g_{\nu\alpha} \cdot L^{\alpha}_{\ \beta}(v) \cdot g^{\beta\rho} = \begin{bmatrix} \gamma & 0 & 0 & -\gamma \cdot v/c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma \cdot \frac{v}{c} & 0 & 0 & \gamma \end{bmatrix}$$

$$= L^{\nu}_{\ \rho}(-v)$$

ii) covariant vectors

$$x_{\nu} = g_{\nu\mu} x^{\mu} = g_{\nu\mu} \cdot L^{\mu}_{\ \rho} \cdot x_o^{\rho} = g_{\nu\mu}\, L^{\mu}_{\ \rho} \cdot g^{\rho\sigma} x_{o\sigma} = L_{\nu}^{\ \sigma} \cdot x_{o\sigma}$$

$\hookrightarrow \boxed{x_{\nu} = L_{\nu}^{\ \sigma} \cdot x_{o\sigma}}$ $\Rightarrow$

$\rightsquigarrow$ and:

$$x_{0\nu} = g_{\nu\mu} \cdot x_0^{\ \mu} = g_{\nu\mu} \cdot L_\rho^{\ \mu} \cdot x^\rho = g_{\nu\mu} \cdot L_\rho^{\ \mu} \cdot g^{\rho\cdot\sigma} \cdot x_\sigma = L_\nu^{\sigma} x_\sigma$$

$$\hookrightarrow \boxed{x_{0\nu} = L_\nu^{\sigma} \cdot x_\sigma}$$

iii) Tensors

$T^\alpha_{\ \rho} = L^\alpha_{\ \nu} \cdot L_\rho^{\ \mu} \cdot T_{0\ \mu}^{\ \nu}$	$T_{0\ \rho}^{\ \alpha} = L_\nu^{\ \alpha} \cdot L^\mu_{\ \rho} \cdot T^\nu_{\ \mu}$
$F_{\alpha\rho} = L_\alpha^{\ \nu} \cdot L_\rho^{\ \mu} F_{0\nu\mu}$ $\downarrow$ and so on!	$F_{0\alpha\rho} = L^\nu_{\ \alpha} \cdot L^\mu_{\ \rho} \cdot F_{0\nu\mu}$ $\downarrow$ an so on!

$\hookrightarrow$ 2. Special Relativity

Now let's apply the 4-D vectors with the transform behavior in 1.) to a physical system.

We define τ as the time the "static-frame"

location: $x_0^\alpha = (0, 0, 0, c\cdot\tau)$

$$\rightsquigarrow x^\alpha = L^\alpha_{\ \nu}(v) \cdot x_0^\nu = \begin{bmatrix} \gamma & 0 & 0 & \gamma v/c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma v/c & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ c\cdot\tau \end{bmatrix} = \begin{bmatrix} \gamma \cdot v \cdot \tau \\ 0 \\ 0 \\ \gamma \cdot c \cdot \tau \end{bmatrix}$$

$\Rightarrow$

__velocity:__ $$w_0^\alpha = \frac{dx_0^\alpha}{d\tau} = (0, 0, 0, c)$$

$$\hookrightarrow w^\alpha = L^\alpha_\nu w_0^\nu = \ldots = (\gamma \cdot v, 0, 0, \gamma \cdot c)$$

$$= \frac{dx^\alpha}{d\tau}$$

__momentum:__ $$p_0^\alpha = m_0 \cdot w_0^\alpha$$

$$p^\alpha = L^\alpha_\nu p_0^\nu = (\gamma \cdot m_0 \cdot v, 0, 0, \gamma \cdot m_0 \cdot c)$$

$$= (\gamma m_0 \cdot v, 0, 0, E/c)$$

with $\boxed{E = \gamma m_0 c^2}$

__Force:__ $$F^\nu := \frac{dp^\nu}{d\tau} = \frac{dp^\nu}{dt} \cdot \frac{dt}{d\tau}$$

$$= \gamma \cdot \frac{dp^\nu}{dt}$$

$$\hookrightarrow F^\nu = \left\{ \gamma \cdot \frac{d}{dt}(m_0 \cdot v \cdot \gamma);\ 0, 0, \gamma \cdot \frac{d}{dt}\left(\frac{E}{c}\right) \right\}$$

$\Longrightarrow$

3. Potentials and Current density as 4D vectors

To show that the Maxwell's equations are invariant against the Lorentz transformation, we have to formulate appropriate 4D-vectors.

This can be done with help of the electrodynamic potentials φ and $\vec{A}$; The Maxwell equation expressed in the potentials are:

Ampere-Maxwell: $$\left[\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{A} = -\mu_0 \vec{j}$$ in vacuum

Poisson: $$\left[\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\varphi = -\frac{1}{\varepsilon_0}\varrho$$

where $\vec{A}, \varphi$ and $\vec{j}, \varrho$ where connected via the Lorentz convention and the continuity equation:

Lorentz convention: $$\boxed{\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0}$$

Continuity equation: $$\boxed{\nabla \cdot \vec{j} + \frac{\partial \varrho}{\partial t} = 0}$$

$\Longrightarrow$

→ in the next step we define the co- and contravariant gradients according to

$$\partial_\nu := \frac{\partial}{\partial x^\nu} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{1}{c} \cdot \frac{\partial}{\partial t} \right)$$

$$\partial^\nu := \frac{\partial}{\partial x_\nu} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, -\frac{1}{c} \cdot \frac{\partial}{\partial t} \right)$$

with this ∂_ν transforms like a covariant and ∂^ν like a contravariant vector.

Also, the expression $\partial_\nu A^\nu$ is Lorentz-invariant, if A^ν transforms like a contrainvariant vector.

<u>Proof:</u> a) $x_0^\mu = L_\varsigma{}^\mu x^\varsigma$

$$\hookrightarrow \frac{\partial x_0^\mu}{\partial x^\nu} = L_\nu{}^\mu$$

$$\rightsquigarrow \partial_\nu = \frac{\partial}{\partial x^\nu} = \frac{\partial x_0^\mu}{\partial x^\nu} \cdot \frac{\partial}{\partial x_0^\mu} = L_\nu{}^\mu \partial_{0\mu}$$

b.) analogous to a.) we get

$$\partial^\nu = L^\nu{}_\mu \cdot \partial_0^\mu$$

c.) $\partial_\nu A^\nu = \partial_\nu L_\varsigma^\nu A_0^\varsigma = L_\varsigma^\nu \partial_\nu A_0^\varsigma$

$$= L_\varsigma^\nu \cdot L_\nu{}^\mu \cdot \partial_{0\mu} A_0^\varsigma = \delta_\varsigma^\mu \partial_{0\mu} A_0^\varsigma$$

$$= \partial_{0\varsigma} A_0^\varsigma$$

⇒

Now, we can formulate a 4D-potential A_μ and a 4D-current S_ν :

$$\boxed{A_\mu := \left(A_1,\ A_2,\ A_3,\ -\frac{1}{c}\varphi \right)}$$

and

$$\boxed{S_\nu := \left(j_1,\ j_2,\ j_3,\ -c \cdot \rho \right)}$$

with which we can rewrite the continuity equation and Lorentz convention:

$$\boxed{\partial^\nu S_\nu = 0} \qquad \text{Continuity equation}$$

$$\boxed{\partial^\nu A_\nu = 0} \qquad \text{Lorentz - convention}$$

Further, the expression

$$\partial^\nu \partial_\nu = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

$$= \underline{\underline{\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}}}$$

is invariant against Lorentz transformation.

The Ampere-Maxwell and Poisson equations rewrite to

$$\boxed{\partial^\nu \partial_\nu A_\mu = -\mu_0 S_\mu}$$

$\Rightarrow$

We still have to prof that A_μ and S_μ transform as covariant vectors:

Im 3-D space we had: $\vec{j} = \rho \cdot \vec{v}$

In analogy, we define a 4D-charge density in the 'rest'-frame as $S_0^\nu = (0, 0, 0, c \cdot \rho_0)$ [ρ_0 := charge density in rest-frame]

$\hookrightarrow S_0^\nu = \rho_0 \cdot \omega_0^\nu$

Applying the Lorentz-transformation, we get

$$S^\nu = \mathcal{L}^\nu_\mu \cdot S_0^\nu = \mathcal{L}^\nu_\mu \rho_0 \omega_0^\mu = \rho_0 \mathcal{L}^\nu_\mu \omega_0^\mu$$

$$\hookrightarrow S^\nu = \rho_0 \cdot \omega^\nu = (\gamma \rho_0 \cdot v, 0, 0, \gamma \cdot \rho_0 \cdot c)$$

With this, S^ν transforms as ω^ν and is therefore a contravariant vector!

If we introduce/define the charge density in the lab-frame as

$$\rho = \gamma \cdot \rho_0 = \frac{\rho_0}{\sqrt{1 - v^2/c^2}}$$

[The charge density changes – but the charge itself remains!]

than the 4D-current density is

$$S^\nu = (\rho \cdot v, 0, 0, \rho \cdot c) \qquad \Rightarrow$$

⇝ The charge density depend on the chosen "Lab-frame" and is <u>not</u> Lorentz-invariant!

This effect is related to the change in volume (length-contraction).

Assume that the volume element in the "rest-frame" is

$$d^3\vec{r}_0 = dx_0\, dy_0\, dz_0 \quad \text{with the rest-charge } dq_0$$

↳ charge density in "rest-frame":

$$\rho_0\, d^3\vec{r}_0 = dq_0$$

If we now observe this charge from the "lab-frame" Σ, than we observe a length-contraction

$$dx = dx_0 \sqrt{1 - v^2/c^2}$$

$$\hookrightarrow d^3\vec{r} = d^3\vec{r}_0 \cdot \sqrt{1 - v^2/c^2}$$

From the principle of charge conservation we get

$$dq = dq_0$$

$$\hookrightarrow \rho_0\, d^3\vec{r}_0 = dq_0 = dq = \rho \cdot d^3\vec{r}_0 \cdot \sqrt{1 - v^2/c^2}$$

$$\hookrightarrow \boxed{\rho_0 = \rho \cdot \sqrt{1 - v^2/c^2}}$$

⇒ The charge q is a <u>conservation quantity</u>. //