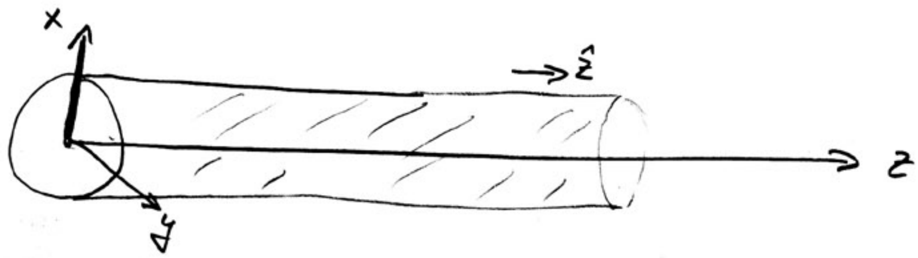
Cylindrical Waveguides, plane waves

Again Maxwell equations: $\nabla \times E = -\frac{\partial \vec{B}}{\partial t} = i\omega \vec{B}$

$$\nabla \times B = \frac{1}{v^2}\frac{\partial E}{\partial t} = -i\varepsilon\mu\omega \vec{E}$$

$$\nabla \cdot E = 0 , \quad \nabla \cdot B = 0$$

↳ Wave equation: $[\Delta + \mu\varepsilon\omega^2]\begin{pmatrix}E\\B\end{pmatrix} = 0$



for wave traveling in z-directions: $\vec{E} = \vec{E}_{(x,y)}\, e^{i(k_0 z - \omega t)}$

$$\boxed{\left[\underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)}_{\Delta_{xy}} + \left(\mu\varepsilon\omega^2 - k^2\right)\right]\begin{pmatrix}E\\B\end{pmatrix} = 0}$$

two-dimensional DE

Next: split B, E-field components in ∥ and transverse components

$$\vec{E} = \vec{E}_z + \vec{E}_t = E_z \cdot \hat{z} + (\hat{z} \times \vec{E}) \times \hat{z}$$

$$\vec{B} = \vec{B}_z + \vec{B}_t = B_z \cdot \hat{z} + (\hat{z} \times \vec{B}) \times \hat{z}$$

↳ Maxwell equ.: $\frac{\partial E_t}{\partial z} + i\omega\, \hat{z} \times B_t = \nabla_t E_z ; \quad \nabla_t = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0\right)$

$$\frac{\partial B_t}{\partial z} - i\mu\varepsilon\omega\, \hat{z} \times \vec{E}_t = \nabla_t B_z$$

⟹

$$\hat{z}\cdot(\nabla_t \times \vec{E}_t) = i\omega B_z\,, \quad \hat{z}\cdot(\nabla_t \times \vec{B}_t) = -i\mu\varepsilon\omega E_z$$

$$\nabla_t \cdot \vec{E}_t = -\frac{\partial E_z}{\partial z}\,, \quad \nabla_t \cdot \vec{B}_t = -\frac{\partial B_z}{\partial z}$$

↳ **Solutions:** $E_t = \frac{i}{\mu\varepsilon\omega^2 - k^2}\left[k\cdot \nabla_t E_z - \omega\,\hat{z}\times\nabla_t B_z\right]$

$$B_t = \frac{i}{\mu\varepsilon\omega^2 - k^2}\cdot\left[k\cdot \nabla_t B_z + \mu\varepsilon\omega\,\hat{z}\times\nabla_t E_z\right]$$

<u>Transverse electromagnetic wave</u> (TEM-wave):

* only field-components transverse to propagation direction

$E_z = 0\,,\ B_z = 0 \rightsquigarrow \boxed{E_t = E_{TEM}}$

with $\boxed{\nabla_t \times E_{TEM} = 0}$ and $\boxed{\vec{\nabla}_t \cdot E_{TEM} = 0}$

↳ TEM is solution of electrostatic problem in 2-dim (x-y-plane).

Wave vector $k = k_0 = \frac{\omega}{\sqrt{\mu\varepsilon}}$

$$B_{TEM} = \sqrt{\mu\varepsilon}\ \hat{z}\times\vec{E}_{TEM}$$

<u>TM-wave:</u> transverse electromagnetic wave

additional boundary value: $E_z\big|_{Surface} = 0 \quad (B_z = 0)$

while for TE wave: $\frac{\partial B_z}{\partial n}\Big|_{Surface} = 0\,,\ (E_z = 0)$

$\Longrightarrow$

Wave Impedance $\mathbf{Z}$:

found via $H_t = \frac{1}{Z} \hat{z} \times \vec{E}_t$

$$\text{with } Z = \begin{cases} k/\varepsilon\cdot\omega = k/k_0 \cdot \sqrt{\mu/\varepsilon} & \text{for TM-wave} \\ \frac{\mu\omega}{k} = k_0/k \cdot \sqrt{\frac{\mu}{\varepsilon}} & \text{for TE wave} \end{cases}$$

$$(k_0 = \omega \cdot \sqrt{\mu\varepsilon})$$

Rewrite solutions E_t, B_t (page 65) for TE and TM waves

$$E_t = \underbrace{\frac{i}{\mu\varepsilon\omega^2 - k^2}}_{\gamma^2} k \cdot \nabla_t E_z = \frac{ik}{\gamma^2} \nabla_t \psi \quad \Big| \; \psi \cdot e^{ikz} = E_z \, (H_z)$$

$$H_t = \frac{ik}{\gamma^2} \nabla_t \psi$$

with ψ has to satisfy $\Big[\underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)}_{\Delta_t} + \gamma^2 \Big] \psi = 0$

with boundary condition $\psi|_S = 0$ (TM) or $\frac{\partial \psi}{\partial n}\Big|_S = 0$ (TE)

γ^2 has to be ≥ 0

↳ dimension of waveguide generates discrete eigenvalues γ_λ^2 corresponding to ψ_λ, $\lambda = 1, 2, \ldots$ ⟹ modes of guide

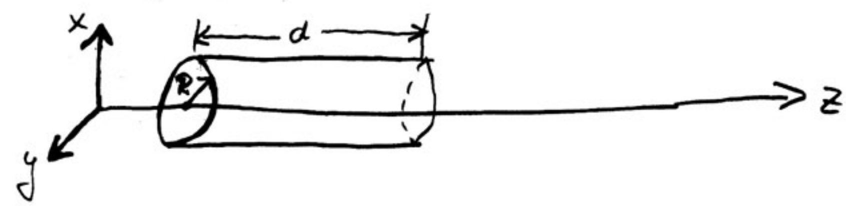
For a given frequ. ω: $k_\lambda^2 = \mu\varepsilon\omega^2 - \gamma_\lambda^2$ with cutoff frequ. $\omega_\lambda = \frac{\gamma_\lambda}{\sqrt{\mu\varepsilon}}$

↳ $k_\lambda = \sqrt{\mu\varepsilon} \cdot \sqrt{\omega^2 - \omega_\lambda^2}$, [$\omega > \omega_\lambda$: k_λ - real ⇝ wave can propagate]

phase velocity $v_p = \frac{\omega}{k_\lambda} = \frac{1}{\sqrt{\mu\varepsilon}} \cdot \frac{1}{\sqrt{1 - (\omega_\lambda/\omega)^2}} > \frac{1}{\sqrt{\mu\varepsilon}}$

Resonant Cavities

Cylindrical cavity with plane end caps



- walls of cavity has infinite conductivity
- inside cavity is filled with lossless dielectric ε, μ

↳ Reflection on end surfaces (z-direction) generates standing waves:

$$A \cdot \sin(k \cdot z) + B \cos(k \cdot z)$$

Boundary condition: 1^{st}-cap @ $z=0$, 2^{nd}-cap @ $z=d$

$$\hookrightarrow \quad k = p \cdot \frac{\pi}{d} \quad (p = 0, 1, \dots)$$

For TM-field ($E_t|_{z=0} = 0$ and $E_t|_{z=d} = 0$):

$$E_z = \psi(x,y) \cdot \cos\left(\frac{p \cdot \pi \cdot z}{d}\right) \qquad p = 0, 1, 2 \dots$$

Similarly for TE-field ($H_z|_{z=0} = 0$; $H_z|_{z=d} = 0$)

$$\hookrightarrow H_z = \psi(x,y) \cdot \sin\left(\frac{p\pi}{d} \cdot z\right) \qquad p = 0, 1, 2, \dots$$

The original solution: $E_t = \frac{i}{\gamma^2} k \nabla_t \psi \rightsquigarrow E_t = -\frac{p\pi}{d\gamma^2} \sin\left(\frac{p\pi z}{d}\right) \nabla_t \psi$
(TM-field)

$$H_t = \frac{i\varepsilon\omega}{\gamma^2} \cos\left(\frac{p\pi}{d} z\right) \cdot \hat{z} \times \nabla_t \psi$$

⟹

$\rightsquigarrow$ and for TE-fields:

$$E_t = -\frac{i\omega\mu}{\gamma^2}\sin\left(\frac{p\pi}{d}\cdot z\right)\cdot\left[\hat{z}\times\nabla_t\psi\right]$$

$$H_t = \frac{p\pi}{d\gamma^2}\cos\left(\frac{p\pi}{d}\cdot z\right)\cdot\nabla_t\psi$$

which satisfy the boundary conditions at the ends of the cavity.

Still to be solved: Eigenvalue problem for ψ:

$$\left[\nabla_t^2+\gamma^2\right]\psi=0$$

with γ^2 is now: $\gamma^2 = \mu\epsilon\omega^2 - \left(\frac{p\pi}{d}\right)^2$ | before: $\mu\epsilon\omega^2 - k^2$

For each value of p, γ_λ^2 determines an eigenfrequ. $\omega_{\lambda p}$ = Resonance frequencies

with $\omega_{\lambda p} = \frac{1}{\mu\cdot\epsilon}\left[\gamma_\lambda^2 + \left(\frac{p\cdot\pi}{d}\right)^2\right]$

and the corresponding fields of that mode.

$\hookrightarrow$ Resonance frequencies form discrete set:

demanding that $k = p\cdot\pi/d$

Find ψ for cylinder with radius R for a TM-mode:

$\rightsquigarrow$ $\psi = E_z$ subject to boundary condition $E_z = 0$ at $\rho = R$

$\hookrightarrow$ $\psi(\rho,\phi) = E_0\, J_m(\gamma_{mn}\rho)\, e^{\pm im\phi}$

with $\gamma_{mn} = \frac{x_{mn}}{R}$ $\Rightarrow$

x_{mn} is the n-th root of $J_m(x) = 0$

↳ Bessel function (page 114, Jackson)

$$\begin{bmatrix} m = 0, 1, 2, \ldots \\ n = 1, 2, 3, \ldots \end{bmatrix}$$

↳ resonance frequencies $\omega_{mnp} = \frac{1}{\sqrt{\mu\varepsilon}} \cdot \sqrt{\frac{x_{mn}^2}{R^2} + \frac{p^2\pi^2}{d^2}}$

lowest TM-mode:

$$m = 0,\ n = 1,\ p = 0 : \ TM_{010} \rightsquigarrow \omega_{010} = \frac{2.405}{\sqrt{\mu\varepsilon} \cdot R}$$

↳ fields $E_z = E_0 \cdot J_0\left(\frac{2.405}{\sqrt{\mu\varepsilon} \cdot R}\right) \cdot e^{-i\omega t}$

$$H_\phi = -i \sqrt{\frac{\varepsilon}{\mu}}\ E_0\ J_1\left(\frac{2.405}{\sqrt{\mu\varepsilon} \cdot R}\right) e^{-i\omega t}$$

The resonance frequency for this mode is independent of <u>d</u>. $\rightsquigarrow$ tuning is not possible!

Solution of ψ for TE-mode:

$\psi = H_z$ subject to boundary condition on H_z: $\left.\frac{\partial \psi}{\partial \rho}\right|_R = 0$

↳ same solution as before

$$\psi(\rho, \phi) = E_0\ J_m(\gamma_{mn}\, \rho)\, e^{\pm i m \phi}$$

$$\gamma = \frac{x'_{mn}}{R}$$

x'_{mn} is n-th root of $J'_m(x) = 0$ $\longrightarrow$

Roots of $J_m'(x)=0$

$m=0 \quad \rightsquigarrow x'_{0n} = 3.832,\ 7,016,\ 10.173$

... see Jackson p. 370

↳ resonance frequencies

$$\omega_{mnp} = \frac{1}{\sqrt{\mu\varepsilon}} \left(\frac{x'^2_{mn}}{R^2} + \frac{p^2\pi^2}{d^2} \right)^{1/2} \quad \Bigg| \begin{array}{l} m=0,1,2\ldots \\ n,p=1,2,3,\ldots \end{array}$$

lowest TE-mode $m=n=p=1$: TE_{111}

$$\rightsquigarrow \omega_{111} = \frac{1.841}{\sqrt{\mu\varepsilon}\cdot R} \cdot \left(1 + 2.912\ \frac{R^2}{d^2} \right)^{1/2}$$

↳ fields:

$$\Psi = H_z = H_0\, J_1\left(\frac{1.841\,\rho}{R} \right) \cdot \cos\phi\ \sin\left(\frac{\pi}{d}\cdot z\right) e^{-i\omega t}$$

For d large enough : $d > 2.03\,R$

$\omega_{111} < \omega_{010}$ (TM-mode)

$\rightsquigarrow TE_{111}$ is fundamental oscillation in cavity

since $\omega_{mnp}\big|_{TE}$ depends on ratio $\frac{d}{R}$ $\rightsquigarrow$ tuning is possible!

Power loss in a Cavity (Q of cavity)

Wave guides & Resonant Cavity have discrete oscillation frequency defined by geometry of guide. ⇝ resonance frequencies (ω_j)

A wave with frequency $\omega \neq \omega_j$ will experience powerloss by dissipation of energy in cavity wall and/or dielectric.

A meassure ~~as~~ $Q := \omega_0 \cdot \frac{\langle \text{Stored Energy} \rangle_t}{\langle \text{Energy loss per cycle} \rangle_t}$

Power dissipation → in ohmic loss = − time rate of change of stored energy U

$$\hookrightarrow \frac{dU}{dt} = -\frac{\omega_0}{Q} U$$

$$\hookrightarrow U(t) = U_0 \, e^{-\frac{\omega_0 t}{Q}}$$ which is the damping of EM-wave

$$E(t) = E_0 e^{-\omega_0 t/Q} \cdot e^{-i(\omega_0 + \Delta\omega)t}$$ (just oscillating part)

This damped oscillation is a superposition of frequencies around $\omega_0 + \Delta\omega$, which can be written as

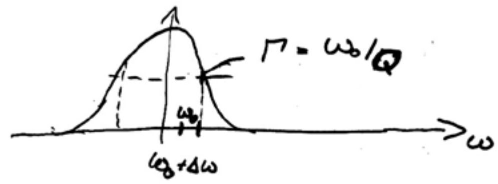
$$E(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E(\omega)\, e^{-i\omega t} d\omega$$

$$\hookrightarrow = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} E_0 \, e^{-\omega_0 t/Q} \cdot e^{i(\omega - \omega_0 - \Delta\omega)\cdot t} dt$$

↳ frequency distribution for the energy in cavity

$$| E(\omega)|^2 \sim \frac{1}{(\omega - \omega_0 - \Delta\omega)^2 + (\omega_0/2Q)^2} \quad \Rightarrow$$

$\leadsto$ Resonance shape



$\Gamma :=$ full-width at half maximum

$\hookrightarrow \; Q = \dfrac{\omega_0}{\Gamma} = \dfrac{\omega_0}{\delta\omega} \leftarrow$ frequency separation

To determine Q : - compute $\langle$ energy stored $\rangle_t$
- determine power loss in the walls.

$\hookrightarrow$ Energy stored :

TM-mode : $U = \dfrac{d}{4}\, \epsilon \left[1 + \left(\dfrac{p\pi}{\gamma_\lambda d} \right)^2 \right] \int_A |\psi|^2 da$ | resonant cavity p. 67

TE-mode : $U = \dfrac{d}{4}\, \mu \left[1 + \dfrac{p\pi}{\gamma_\lambda d} \right] \int_A |\psi|^2 da$ | resonant cavity p. 68

The power loss :

$$P_{loss} = \frac{1}{2\sigma\delta} \Big[\oint_C dl \underbrace{\int_0^d dz \, [\hat{n} \times \vec{H}]^2_{sides}}_{\text{length of cavity}} + 2 \underbrace{\int_A da \, [\hat{n} \times \vec{H}]^2_{end}}_{\text{end caps}} \Big]$$

C is circumferenc
A is cross-section
δ - skin depth , σ : conductivity

For TM-mode $p \neq 0$

dimensionless # of the order of unity

$$\leadsto P_{loss} = \frac{\epsilon}{\sigma\delta\mu} \left[1 + \left(\frac{p\pi}{\gamma_\lambda d} \right)^2 \right] \left(1 + \xi_\lambda \cdot \frac{C \cdot d}{4A} \right) \int_A |\psi|^2 da$$

Volume

$$\hookrightarrow Q = \frac{\mu}{\mu_c} \frac{d}{\delta} \frac{1}{2\left(1 + \xi_\lambda \frac{C \cdot d}{4A}\right)} = \underbrace{\frac{\mu}{\mu_c}}_{\text{perm. of metal}} \cdot \left(\frac{V}{S \cdot \delta} \right) \times (\text{Geometrical factor})$$

S: surface area

$\leadsto$ Q of cavity is ratio: $\dfrac{(\text{volume occupied by fields})}{(\text{volume of conductor into which field penetrate})} \times (\text{geometrical factor})$