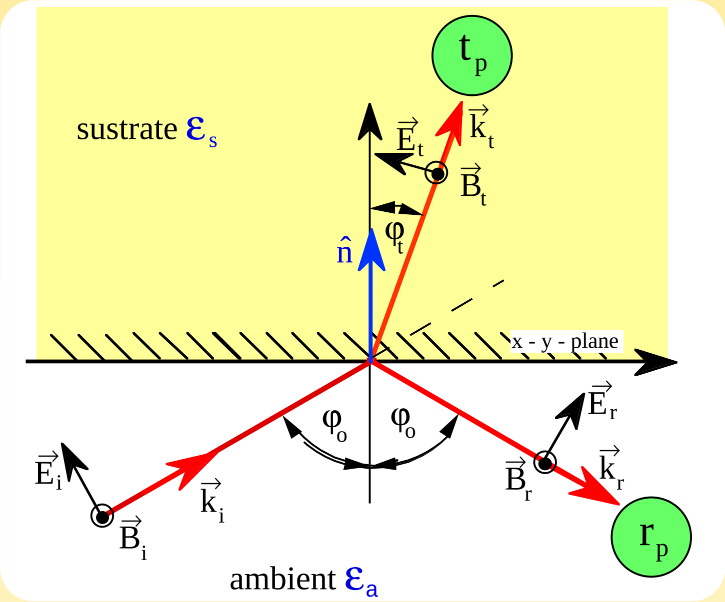
Reflection and Refraction in Multi-layered Heterostructures: Interference

Starting point are the Fresnel's equations derivated from the Maxwell's equations for plane waves on a plane interface between two media. The resulting Fresnel's coefficients for p-/s- polarized light with $\mu_s = \mu_a = 1$ were:

Fresnel's coefficients for p-polarized light:

$$r_p = \frac{\varepsilon_s \cdot \cos\varphi_o - \sqrt{\varepsilon_a} \cdot \sqrt{\varepsilon_s - \varepsilon_a \cdot \sin^2\varphi_o}}{\varepsilon_s \cdot \cos\varphi_o + \sqrt{\varepsilon_a} \cdot \sqrt{\varepsilon_s - \varepsilon_a \cdot \sin^2\varphi_o}}$$

$$t_p = \frac{2 \cdot \sqrt{\varepsilon_a \cdot \varepsilon_s} \cos\varphi_o}{\varepsilon_s \cdot \cos\varphi_o + \sqrt{\varepsilon_a} \sqrt{\varepsilon_s - \varepsilon_a \cdot \sin^2\varphi_o}}$$



Fresnel's coefficients for s-polarized light:

$$r_s = \frac{\sqrt{\varepsilon_a} \cdot \cos\varphi_o - \sqrt{\varepsilon_s - \varepsilon_a \cdot \sin^2\varphi_o}}{\sqrt{\varepsilon_a} \cdot \cos\varphi_o + \sqrt{\varepsilon_s - \varepsilon_a \cdot \sin^2\varphi_o}}$$

$$t_s = \frac{2 \cdot \sqrt{\varepsilon_a} \cos\varphi_o}{\sqrt{\varepsilon_a} \cdot \cos\varphi_o + \sqrt{\varepsilon_s - \varepsilon_a \cdot \sin^2\varphi_o}}$$

- For a multi-layered heterostructure we have to sum over each reflected- and transmitted component at each interface to obtain the total reflected and transmitted wave component!

Reflection and Refraction in Multi-layered Heterostructures: Interference

Assume a multi-layered medium built up by a stack of k-interfaces formed by "k-1" isotropic layers on top of a isotropic susbtrate.

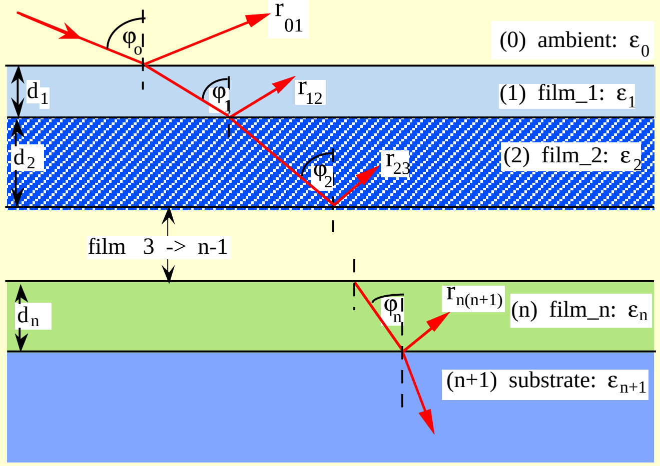
Start labeling from "0" ambient; "1" first layer; …. "k-1" k^{th}-layer and "k": interface k^{th}-layer to substrate. The Fresnel's coefficients for a general interface is given through:

$$r_{p}\big|_{k(k+1)} = \frac{\varepsilon_{k+1}\cdot\sqrt{\varepsilon_k - \varepsilon_o\cdot\sin^2\varphi_o} - \varepsilon_k\cdot\sqrt{\varepsilon_{k+1} - \varepsilon_o\cdot\sin^2\varphi_o}}{\varepsilon_{k+1}\cdot\sqrt{\varepsilon_k - \varepsilon_o\cdot\sin^2\varphi_o} + \varepsilon_k\cdot\sqrt{\varepsilon_{k+1} - \varepsilon_o\cdot\sin^2\varphi_o}}$$

$$r_{s}\big|_{k(k+1)} = \frac{\sqrt{\varepsilon_k}\cdot\cos\varphi_o - \sqrt{\varepsilon_{k+1} - \varepsilon_o\cdot\sin^2\varphi_o}}{\sqrt{\varepsilon_k}\cdot\cos\varphi_o + \sqrt{\varepsilon_{k+1} - \varepsilon_o\cdot\sin^2\varphi_o}}$$

$$t_{p}\big|_{k(k+1)} = \frac{2\cdot\sqrt{\varepsilon_{k+1}}\cdot\sqrt{\varepsilon_k - \varepsilon_o\cdot\sin^2\varphi_o}}{\varepsilon_{k+1}\cdot\cos\varphi_o + \sqrt{\varepsilon_k}\sqrt{\varepsilon_{k+1} - \varepsilon_o\cdot\sin^2\varphi_o}}$$

$$t_{s}\big|_{k(k+1)} = \frac{2\cdot\sqrt{\varepsilon_k - \varepsilon_o\cdot\sin^2\varphi_o}}{\sqrt{\varepsilon_k - \varepsilon_o\cdot\sin^2\varphi_o} + \sqrt{\varepsilon_{k+1} - \varepsilon_o\cdot\sin^2\varphi_o}} \qquad (k \geq 0)$$

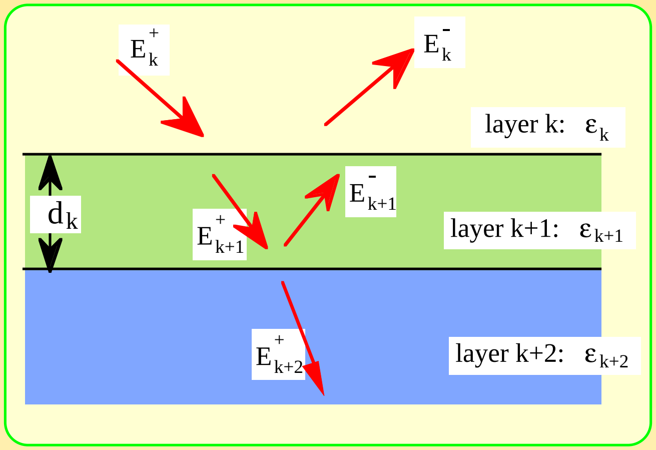


The change in the phase of the wave traversing through the k^{th}-layer is given by:

$$\Phi_k = \frac{2\,\pi\,d_k}{\lambda}\sqrt{\varepsilon_k - \varepsilon_0\,\sin^2\varphi} \qquad (k \geq 1)$$

Reflection and Refraction in Multi-layered Heterostructures: Matrix method

Consider a wave incident on an interface of index k(k+1) between media of dielectric functions ε_k, ε_{k+1} and suppose the amplitude of the electric field vector is E_k^+. That of the reflected wave is E_k^-. Inside the film (ε_{k+1}) the resultant of all positive-going waves sum to E_{k+1}^+ and those negative-going to E_{k+1}^-.



A linear correlation between layer k and (k+1) may be written in matrix form:

$$\begin{bmatrix} E_k^+ \\ E_k^- \end{bmatrix} = \frac{1}{t_{k+1}} \begin{bmatrix} 1 & r_{k(k+1)} \\ r_{k(k+1)}\, e^{-2i\Phi_{k-1}} & e^{-2i\Phi_{k-1}} \end{bmatrix} \begin{bmatrix} E_{k+1}^+ \\ E_{k+1}^- \end{bmatrix} = \frac{1}{t_{k+1}} [M_{k+1}] \begin{bmatrix} E_{k+1}^+ \\ E_{k+1}^- \end{bmatrix}$$

with

$$\begin{bmatrix} E_o^+ \\ E_o^- \end{bmatrix} = \frac{[M_1]\cdot[M_2]\cdot \;\dots\; \cdot[M_{k+1}]}{t_1 \cdot t_2 \cdot \;\dots\; \cdot t_{k+1}} \begin{bmatrix} E_{k+1}^+ \\ E_{k+1}^- \end{bmatrix}$$

For the first interface (k = 0): $\Rightarrow \quad M_1 = \begin{bmatrix} 1 & r_{01} \\ r_{01} & 1 \end{bmatrix}$ (with $e^{-2i\Phi_o} = 1$, since $\Phi_o = 0$)

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Reflection and Refraction in Multi-layered Heterostructures: Matrix method

For a system with 'k' layers we have to form the matrix product of "k" times [2•2] matrix multiplication:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & r_{01} \\ r_{01} & 1 \end{bmatrix} \times \begin{bmatrix} 1 & r_{12} \\ r_{12}\, e^{-2i\,\Phi_1} & e^{-2i\,\Phi_1} \end{bmatrix} \times \begin{bmatrix} 1 & r_{23} \\ r_{23}\, e^{-2j\Phi_2} & e^{-2i\,\Phi_2} \end{bmatrix} \times \ldots \times \begin{bmatrix} 1 & r_{(k-1)k} \\ r_{(k-1)k}\, e^{-2i\,\Phi_k} & e^{-2i\,\Phi_k} \end{bmatrix}$$

to obtain the complex reflectance and transmission amplitudes from:

$$rr = \frac{E_o^-}{E_o^+} = \frac{c}{a}; \quad \text{and} \quad tt = \frac{E_{k+1}^+}{E_o^+} = \frac{t_1 \cdot t_2 \cdot \ \ldots \ \cdot t_{k+1}}{a};$$

The reflectivity and transmittance are given by:

$$R = rr \cdot rr^* = \frac{c \cdot c^*}{a \cdot a^*}; \quad \text{and} \quad T = tt \cdot tt^* = \frac{(t_1 \cdot t_2 \ \ldots \ t_{k+1})(t_1^* \cdot t_2^* \ \ldots \ t_{k+1}^*)}{a \cdot a^*};$$

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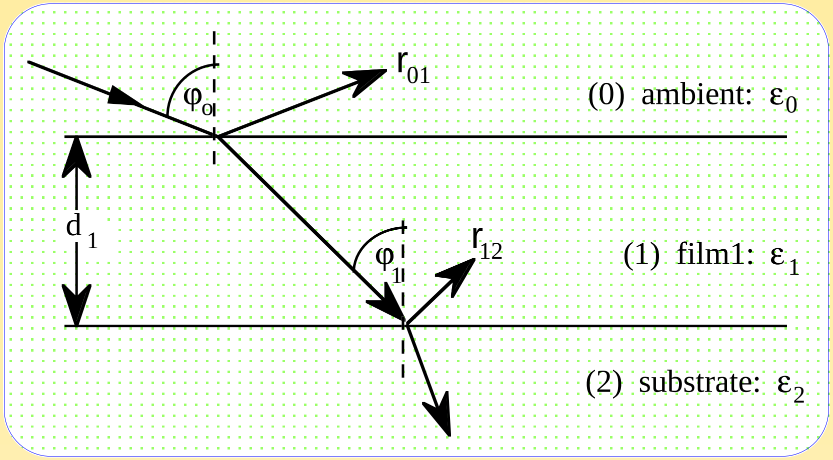
Reflection amplitudes in multi-layered Heterostructures: 3- / 4-layered stacks

Reflectance amplitude for a 3-layer stack:

$$rr_{012} = \frac{r_{01} + r_{12}\, e^{-2i\Phi_1}}{1 + r_{01}\, r_{12}\, e^{-2i\Phi_1}}$$

with the phase factor:

$$\Phi_1 = \frac{2\pi d_1}{\lambda} \sqrt{\varepsilon_1 - \varepsilon_0 \sin^2\varphi}$$



Reflectance amplitude for a 4-layer stack:

$$rr_{0123} = \frac{r_{01} + r_{12}\, e^{-2i\Phi_1} + r_{23}\, e^{-2i(\Phi_1+\Phi_2)} + r_{01}\, r_{12}\, r_{23}\, e^{-2i\Phi_2}}{1 + r_{01}\, r_{12}\, e^{-2i\Phi_1} + r_{01}\, r_{23}\, e^{-2i(\Phi_1+\Phi_2)} + r_{12}\, r_{23}\, e^{-2i\Phi_2}}$$

with the phase factors:

$$\Phi_1 = \frac{2\pi d_1}{\lambda} \sqrt{\varepsilon_1 - \varepsilon_0 \sin^2\varphi}$$

and

$$\Phi_2 = \frac{2\pi d_2}{\lambda} \sqrt{\varepsilon_2 - \varepsilon_1 \sin^2\varphi} \,.$$