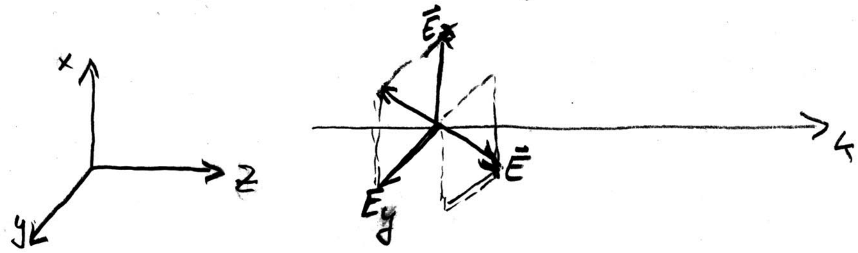
Polarization

- light is treated as transverse, electromagnetic wave
- So far we considered linearly - or in plane polarized light.



linear Polarization:

$$E_x(z,t) = \hat{e}_x \cdot E_{0x} \cdot \cos(k \cdot z - \omega \cdot t)$$
$$E_y(z,t) = \hat{e}_y \cdot E_{0y} \cdot \cos(k z - \omega \cdot t + \underset{\substack{\uparrow \\ \text{phase} \\ \text{shift}}}{\varepsilon})$$

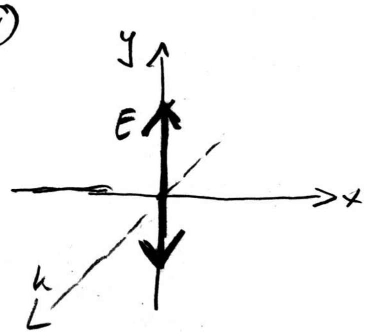
exponentional:

$$E_x = E_{0x}\, e^{i(kz - \omega t + \varphi_x)}$$
$$E_y = E_{0y}\, e^{i(k \cdot z - \omega t + \varphi_y)}$$

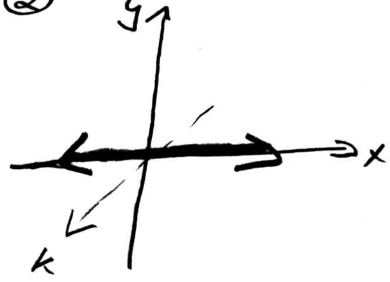
$$\hookrightarrow \quad \vec{E} = \hat{e}_x E_{0x} \cdot e^{i(kz - \omega t + \varphi_x)} + \hat{e}_y E_{0y}\, e^{i(kz - \omega t + \varphi_y)}$$

$$= \underbrace{\left[\hat{e}_x E_{0x} e^{i\varphi_x} + \hat{e}_y E_{0y} e^{i\varphi_y}\right]}_{\tilde{E}_0 = \begin{bmatrix} E_{0x} e^{i\varphi_x} \\ E_{0y} e^{i\varphi_y} \end{bmatrix}} \cdot e^{i(kz - \omega t)} = \tilde{E}_0\, e^{i(kz - \omega t)}$$

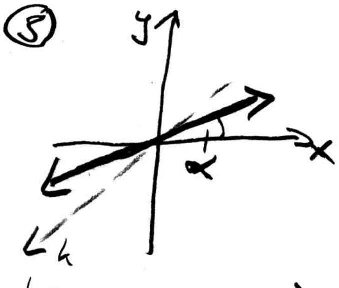
$:=$ two-dim complex amplitude
= "Jones vector"

① 

$$\tilde{E}_0 = \begin{bmatrix} 0 \\ A \end{bmatrix} = A \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

② 

$$\tilde{E}_0 = \begin{bmatrix} A' \\ 0 \end{bmatrix} = A' \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

③ 

$$\tilde{E}_0 = \begin{bmatrix} E_{0x} e^{i\varphi_x} \\ E_{0y} e^{i\varphi_y} \end{bmatrix} = \begin{bmatrix} A \cdot \cos\alpha \\ A \cdot \sin\alpha \end{bmatrix} = A \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$$

Polarization

contin. linear Polarization

③ $\tilde{E}_o = A \cdot \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$, for normalized vector $\tilde{E}_o$: $A = 1$ since $\sin^2\varphi + \cos^2\varphi = 1$

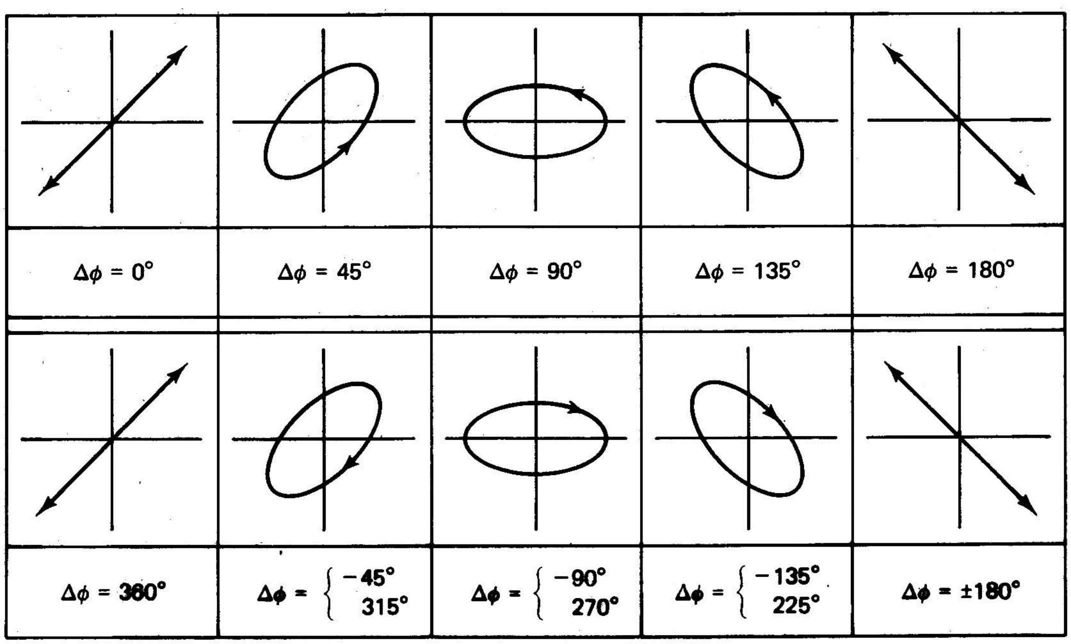
example: $\alpha = 60°$ and $A = 1$,

$$\tilde{E}_o = \begin{bmatrix} \cos(60) \\ \sin(60) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} := \underline{\text{Jones vector!}}$$

General: a Jones vector $\begin{bmatrix} a \\ b \end{bmatrix}$ with $a, b \in \mathbb{R}$, not both zero, represents linearly polarized light at inclination angle $\alpha = \tan^{-1}(b/a)$.

if phase difference between the vibrations is not "0°" or "180°"
$\rightsquigarrow$ resulting $\vec{E}$-vector traces a sequence of Lissajous figures depending on the relative phase difference
$\Delta\varphi = \varphi_y - \varphi_x$ for the general case of $E_{ox} \neq E_{oy}$. $\Rightarrow$ $\boxed{\text{elliptical or circular polarization}}$

$\Rightarrow$



phase lag convention $\Delta\phi = \Delta\varphi = \varphi_y - \varphi_x$

For $E_{ox} = E_{oy}$ $\rightsquigarrow$ $\Delta\varphi = 90°$ or $270°$ reduces to circles!

Polarization

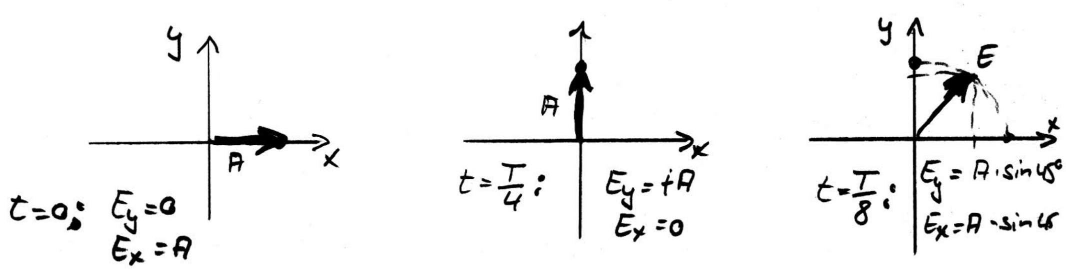
Suppose:

$E_{0x} = E_{0y}$ and E_x leads E_y by $90°$

$$@\, z=0 \leadsto \left.\begin{aligned} E_x &= E_{0x}\, e^{-i\omega t} \\ E_y &= E_{0y}\, e^{-i(\omega t - \frac{\pi}{2})} \end{aligned}\right\} \to \text{real part: } \left[\begin{aligned} E_x &= A \cos\omega t \\ E_y &= A \cdot \cos(\omega t - \tfrac{\pi}{2}) = A \cdot \sin\omega t \end{aligned}\right]$$

with $\omega = 2\pi\nu = \frac{2\pi}{T}$ and $E^2 = E_x^2 + E_y^2 = A^2 \cdot [\cos^2\omega t + \sin^2\omega \cdot t] = A^2$

$\leadsto$



Jones vector:

$$\tilde{E}_0 = \begin{bmatrix} E_{0x}\, e^{i\varphi_x} \\ E_{0y}\, e^{i\varphi_y} \end{bmatrix} = \begin{bmatrix} A \\ A \cdot e^{i\pi/2} \end{bmatrix} = A \cdot \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Normalization: $A^2 \cdot [1^2 + i^2] = 1 \Rightarrow 2 \cdot A = 1 \leadsto$

$$\tilde{E}_0 = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 \\ i \end{bmatrix}$$:= represents left-circularly polarized light. (counterclock wise)

Suppose: $E_{0x} = E_{0y}$ and E_y leads E_x by $90°$

$$\leadsto \tilde{E}_0 = \begin{bmatrix} E_{0x}\, e^{i\varphi_x} \\ E_{0y}\, e^{i\varphi_y} \end{bmatrix} = \begin{bmatrix} A \\ A \cdot e^{-i\pi/2} \end{bmatrix} = \ldots \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

right-circularly polarized light. (clock wise circulation)

Example: Vector $\begin{bmatrix} 2i \\ 2 \end{bmatrix}$ what is the character of light?

$$\begin{bmatrix} 2i \\ 2 \end{bmatrix} = 2 \begin{bmatrix} i \\ 1 \end{bmatrix} = \underbrace{2i}_{\text{affects only amplitude!}} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$ $\leadsto$ right-circularly polarized

Polarization

Suppose: $E_{0x} = A$, $E_{0y} = B$, $A \neq B$

$\rightsquigarrow \begin{bmatrix} A \\ iB \end{bmatrix}$ = counter-clock wise rotation $\qquad \begin{bmatrix} A \\ -iB \end{bmatrix}$ = clock wise rotation

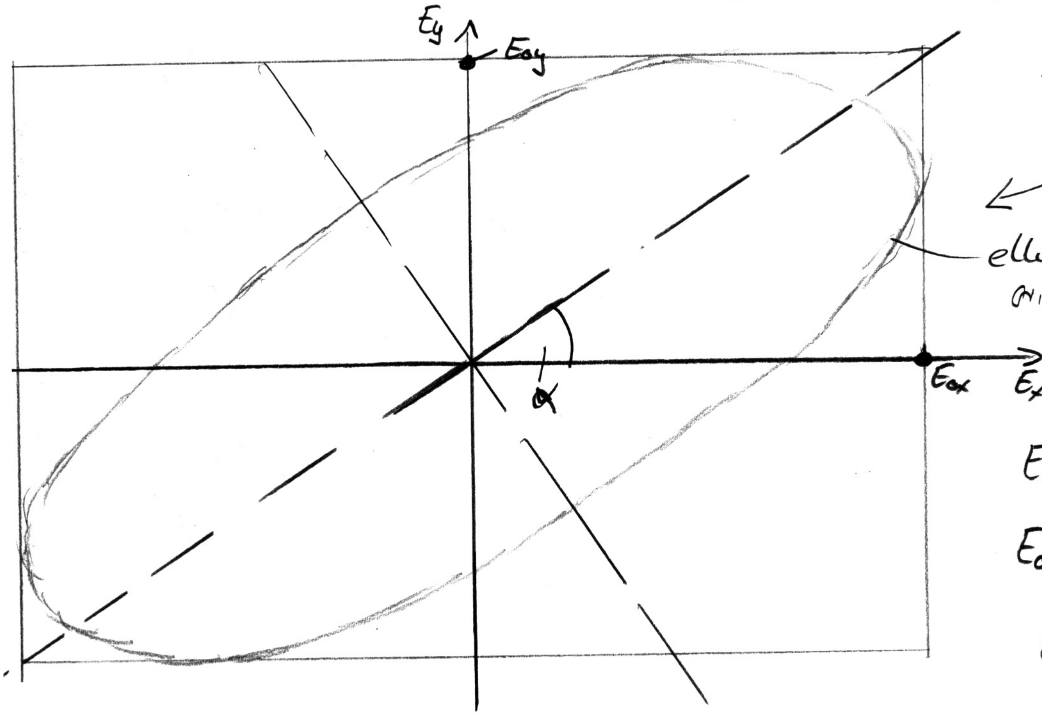
Example: E_x lead E_y by $\varepsilon = \varphi_y - \varphi_x$. Take $\varphi_x = 0$, $\varphi_y = \varepsilon$,
$E_{0x} = A$, $E_{0y} = b$

use Euler theorem

$\rightsquigarrow$ Jones vector: $\tilde{E}_0 = \begin{bmatrix} A \\ b e^{i\varepsilon} \end{bmatrix}$, $b e^{i\varepsilon} = b(\cos\varepsilon + i\sin\varepsilon) = B + iC$

$$\rightsquigarrow \tilde{E}_0 = \begin{bmatrix} A \\ B + iC \end{bmatrix}$$

Normalized vector: $\tilde{E}_0 = \frac{1}{\sqrt{A^2+B^2+C^2}} \begin{bmatrix} A \\ B + iC \end{bmatrix}$



$$\tan 2\alpha = \frac{2 E_{0x} E_{0y} \cos\varepsilon}{E_{0x}^2 - E_{0y}^2}$$

elliptical polarized light oriented at an angle relative to the x-axis.

$E_{0x} = A$

$E_{0y} = \sqrt{B^2 + C^2}$

$\varepsilon = \tan^{-1}\left(\frac{C}{B}\right)$

Polarization

Example: Analyze the Jones vector $\begin{bmatrix} 3 \\ 2+i \end{bmatrix}$ to show that it represent elliptically polarized light.

Solution:

relative phase between φ_y and φ_x

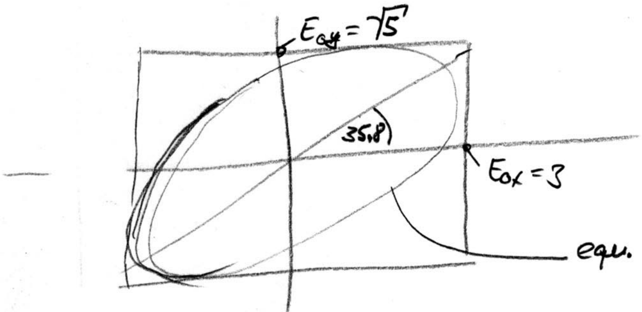
$$\varepsilon = \varphi_y - \varphi_x = \tan^{-1}\left(\frac{C}{B}\right) = \tan^{-1}\left(\frac{1}{2}\right) = 26.6°$$

$$E_{0x} = A = 3$$

$$E_{0y} = \sqrt{B^2 + C^2} = \sqrt{2^2 + 1^2} = \sqrt{5}$$

$$\alpha = \frac{1}{2}\tan^{-1}\left[\frac{2\, E_{0x} \cdot E_{0y} \cdot \cos\varepsilon}{E_{0x}^2 - E_{0y}^2}\right] = \frac{1}{2}\cdot\tan^{-1}\left[\frac{2 \cdot 3 \cdot \sqrt{5} \cdot 26.6°}{3^2 - 5}\right]$$

$$= \underline{\underline{35.8°}}$$



equ. of Ellipe: $\left(\frac{E_x}{E_{0x}}\right)^2 + \left(\frac{E_y}{E_{0y}}\right)^2 - 2\left(\frac{E_x}{E_{0x}}\right)\left(\frac{E_y}{E_{0y}}\right)\cdot\cos\varepsilon = \sin^2\varepsilon$

$$\rightsquigarrow \boxed{\frac{E_x^2}{9} + \frac{E_y^2}{5} - 0.267 \cdot E_x \cdot E_y = 0.2}$$

Superposition of polarized wave:

a.) $\begin{bmatrix} 1 \\ i \end{bmatrix} + \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 1+1 \\ i+(-i) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

left-circul. | right-circul. | linear polarized, double amplitude

b.) $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

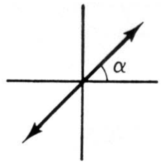
horizontally polar. | vertically polar. | linearly polarized under 45°

There is no Jones vector represent unpolarized or partial polarized light!

Polarization

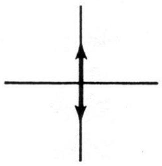
I.) linear

General:

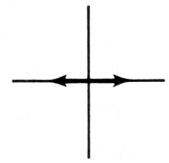


$$E_0 = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$$

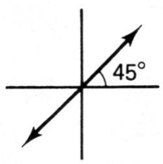
Vertical: $E_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



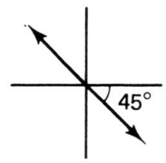
Horizontal: $E_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



At + 45°: $E_0 = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

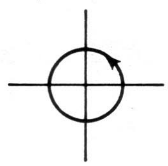


At −45°: $E_0 = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$



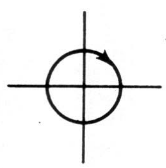
II. Circular Polarization $\left(\Delta\phi = \dfrac{\pi}{2}\right)$

Left:



$$E_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$$

Right:

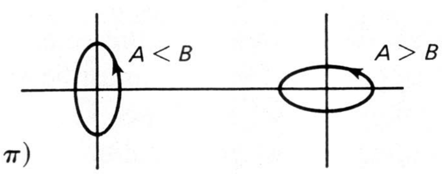


$$E_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

III. Elliptical Polarization

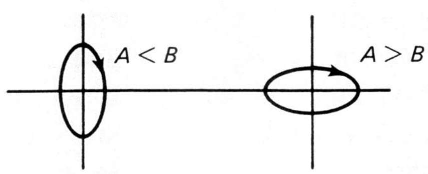
Left:

$(\Delta\phi = (m + \frac{1}{2})\,\pi)$



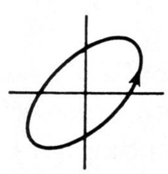
$$E_0 = \frac{1}{\sqrt{A^2 + B^2}}\begin{bmatrix} A \\ iB \end{bmatrix}$$

Right:



$$E_0 = \frac{1}{\sqrt{A^2 + B^2}}\begin{bmatrix} A \\ -iB \end{bmatrix}$$

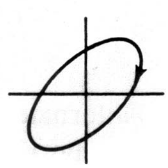
Left:



$$E_0 = \frac{1}{\sqrt{A^2 + B^2 + C^2}}\begin{bmatrix} A \\ B + iC \end{bmatrix}$$

$\left(\Delta\phi \neq \begin{cases} m\pi \\ (m + \frac{1}{2})\pi \end{cases}\right)$

Right:

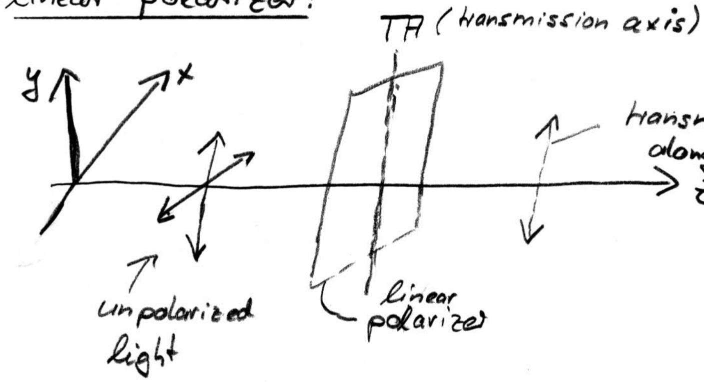


$$E_0 = \frac{1}{\sqrt{A^2 + B^2 + C^2}}\begin{bmatrix} A \\ B - iC \end{bmatrix}$$

Polarization

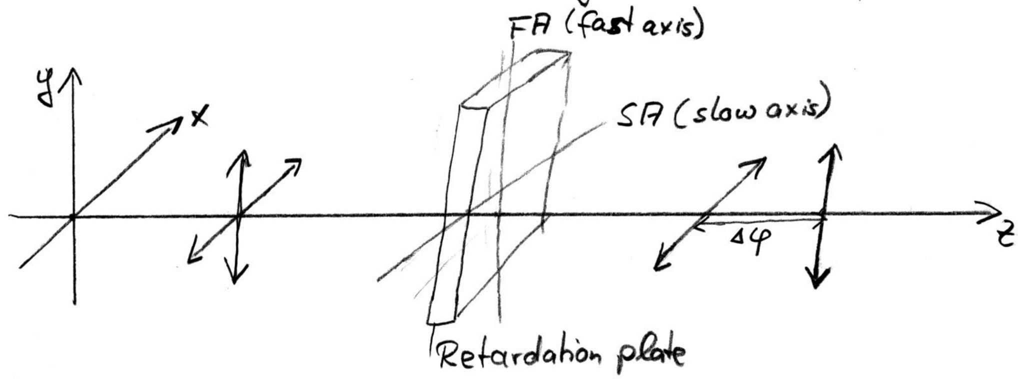
Mathematical Representation of Polarizers

1) linear polarizer:



transmitted light includes only components along y-axis and is therefore linear polarized in the vertical, or y-direction. The process is assumed to be 100%.

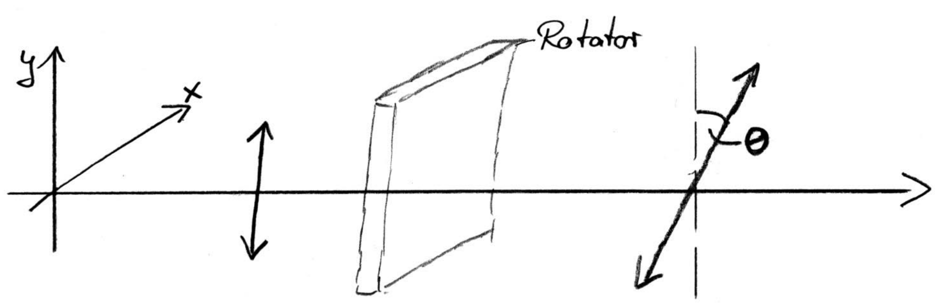
2.) Phase Retarder: introduces a phase difference between two orthogonal E-components



if net phase difference $\Delta\varphi = 90°$ $\rightsquigarrow$ quater-wave plate

" $\Delta\varphi = 180°$ $\rightsquigarrow$ half-wave plate

3.) Rotator: = rotates the direction of linear polarized light incident on it - by some particular angle



Polarization

Matrix formulation: $[2\times 2]$ - matrix, representing the polarizer function

1.) linear vertically (y-direction) polarized light

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightsquigarrow \left.\begin{matrix} a(0) + b\cdot(1) = 0 \\ c(0) + d(1) = 1 \end{matrix}\right\} \Rightarrow M = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\rightsquigarrow$ analog: TA - horizontal $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

TA inclined @ 45°:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\rightsquigarrow a+b = 1, \; a-b = 0, \; c+d = 1, \; c-d = 0$

$\rightsquigarrow a = b = c = d = 1/2$

$\rightsquigarrow M = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ linear polarizer; TA at 45°

linear polarizer, TA at angle θ:

$$M = \begin{bmatrix} \cos^2\theta & \sin\theta\cdot\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

2.) Phase retarders:

matrix has to transform $E_{0x}\, e^{i\varphi_x} \longrightarrow E_{0x}\, e^{i(\varphi_x+\varepsilon_x)}$

and $E_{0y}\, e^{i\varphi_y} \longrightarrow E_{0y}\, e^{i(\varphi_y+\varepsilon_y)}$

$$\rightsquigarrow \underbrace{\begin{bmatrix} e^{i\varepsilon_x} & 0 \\ 0 & e^{i\varepsilon_y} \end{bmatrix}}_{M_{\text{Phase retarder}} = M_{PR}}\begin{bmatrix} E_{0x}\, e^{i\varphi_x} \\ E_{0y}\, e^{i\varphi_y} \end{bmatrix} = \begin{bmatrix} E_{0x}\, e^{i(\varphi_x+\varepsilon_x)} \\ E_{0y}\, e^{i(\varphi_y+\varepsilon_y)} \end{bmatrix}$$

<u>Example</u>: consider quarter-wave plate (QWP) for which $\Delta\varepsilon = \frac{\pi}{2}$

$\varepsilon_y - \varepsilon_x = \frac{\pi}{2}$ (SA vertical) ; $\varepsilon_x - \varepsilon_y = \frac{\pi}{2}$ (SA horizontal)

$\rightsquigarrow M = e^{-i\pi/4}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ $\quad \hookrightarrow M = e^{i\pi/4}\begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$

Polarization

Example: consider half-wave plate (HWP), where $\Delta\varepsilon = \pi$

$\varepsilon_y - \varepsilon_x = \pi$ (SA - vertical)

$$\hookrightarrow M = \begin{bmatrix} e^{-i\pi/2} & 0 \\ 0 & e^{+i\pi/2} \end{bmatrix} = e^{-i\pi/2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\varepsilon_x - \varepsilon_y = \pi$ (SA - horizontal)

$$\hookrightarrow M = \begin{bmatrix} e^{i\pi/2} & 0 \\ 0 & e^{-i\pi/2} \end{bmatrix} = e^{+i\pi/2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

3.) **Rotator**: Rotate E-vector oscillating linearly at angle θ to one that oscillate linearly at angle $(\theta + \beta)$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta+\beta) \\ \sin(\theta+\beta) \end{bmatrix}$$

from
$$\cos(\theta+\beta) = \cos\theta\cdot\cos\beta - \sin\theta\cdot\sin\beta$$
$$\sin(\theta+\beta) = \sin\theta\cdot\cos\beta + \cos\theta\cdot\sin\beta$$

$\hookrightarrow a = \cos\beta$, $b = -\sin\beta$, $c = \sin\beta$, $d = \cos\beta$

$$\hookrightarrow M = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$: rotator through angle $\underline{+\beta}$

$\theta \Rightarrow \theta + \beta$

Polarization

Example:

What is the result of allowing left-circular polarized light to pass through an eight-wave plate?

Solution: left circulate light is represented by $\begin{bmatrix} 1 \\ i \end{bmatrix}$

matrix element for 8-wave plate:

$$\Delta\varphi = \frac{2\pi}{8} = \frac{\pi}{4} \text{ or } 45^\circ$$

let $\varepsilon_x = 0$ $\rightsquigarrow$ $M = \begin{bmatrix} e^{i\varepsilon_x} & 0 \\ 0 & e^{i\varepsilon_y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$

operated on left circulate light:

$$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ i e^{i\pi/4} \end{bmatrix} = \begin{bmatrix} 1 \\ e^{i\frac{\pi}{2}} \cdot e^{i\pi/4} \end{bmatrix} = \begin{bmatrix} 1 \\ e^{i 3/4 \pi} \end{bmatrix}$$

↳ light is elliptically polarized and components are 135° out of phase. Use Euler equation

$$e^{i\varepsilon} = \cos\varepsilon + i\sin\varepsilon \Rightarrow e^{i 3/4\pi} = \cos(3/4\,\pi) + i\sin(3/4\,\pi)$$
$$= -\frac{1}{\sqrt{2}} + i\left(\frac{1}{\sqrt{2}}\right)$$

$$M_{ellip} = \begin{bmatrix} A \\ B+iC \end{bmatrix} \rightsquigarrow A = 1, \quad B = -\frac{1}{\sqrt{2}}, \quad C = \frac{1}{\sqrt{2}}$$

$$\rightsquigarrow E_{ox} = 1, \quad E_{oy} = \sqrt{B^2+C^2} = 1, \quad \varepsilon = \tan^{-1}\left(\frac{C}{B}\right) = -45^\circ$$

$$\alpha = \frac{1}{2}\tan^{-1}\left[\frac{2 E_{ox}\cdot E_{oy}\cdot\cos\varepsilon}{\underbrace{E_{ox}^2 - E_{oy}^2}_{=\phi}}\right] = \lightning = -45^\circ \text{ since } \underbrace{E_{ox} = E_{oy}}_{\text{circular polarized}}$$