D. Maxwell's Equations (Jackson Chapt. VI)

Maxwell realized that Ampere's law cannot be correct in general, since $\boxed{\nabla \cdot \vec{J} = 0}$ only applies to steady current, but not to time-dependent currents.

from $\nabla \times \vec{H} = \vec{J} \rightsquigarrow \nabla \cdot (\nabla \times \vec{H}) = 0 = \nabla \cdot \vec{J} = -\dot{\rho}$

$\rightsquigarrow$ We need a generalized field equation!

Form Poisson equ. $\nabla \cdot \vec{D} = \rho$

and

continuity equ. $\dot{\rho} + \nabla \cdot \vec{J} = 0$

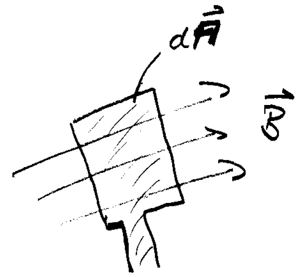
follows $\nabla \cdot \dot{\vec{D}} = \dot{\rho} = -\nabla \cdot \vec{J}$ or $\boxed{\nabla \cdot (\dot{\vec{D}} + \vec{J}) = 0}$

which mean that in the non-stationary case a "displacement current" $\frac{\partial \vec{D}}{\partial t}$ is added to the current charge density $\vec{J}$!

The general field equation becomes

$$\boxed{\nabla \times \vec{H} = \dot{\vec{D}} + \vec{J}}$$ 'Maxwell - Ampere' law

D.1. Faraday's law of Induction



The magnetic flux ϕ generated by a magn. flux density B through an area A is defined as

$$\boxed{\phi = \int_F \vec{B} \cdot d\vec{A}}$$

If we assume a conductor loop enclosing the area F, the induced voltage U is

$$U_{ind} = -\frac{d\phi}{dt} \quad \text{or} \quad = -N \frac{d\phi}{dt}$$

N: # of conductor loops

The time-dependent change in the flux ϕ can be accompl. by either both. varying the magn. flux density B, or changing the enclosed area F.

(a) assume we fix the area F:

from $U = \int_s \vec{E} \cdot d\vec{s}$ [remember $\vec{E} = -\nabla\phi$, $U_{ind} = -\frac{d\phi}{\partial t}$

follows

$$U_{ind} = \int_{\partial F} \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int_F \vec{B} \cdot d\vec{A} = -\int_F \dot{\vec{B}} \cdot d\vec{A}$$

$\parallel$ (Stoke's theorem)

$$\int_F (\nabla \times \vec{E}) \cdot dA$$

$\Longrightarrow$

$$\rightsquigarrow \quad \int_F (\nabla \times \vec{E}) \cdot d\vec{A} = - \int_F \dot{\vec{B}} \cdot d\vec{A}$$

$$\text{or} \quad \int_F \left[\nabla \times \vec{E} + \frac{d\vec{B}}{dt} \right] \cdot d\vec{A} = 0$$

Since integral has to vanish for any fixed area F, we get

$$\boxed{\nabla \times \vec{E} = - \dot{\vec{B}}}$$

differential form of Faraday's law of induction.

D.2. Maxwell's equation in matter

So far we derived <u>four</u> field equation and three materials equations – which all together – are denoted as Maxwell's equations:

a) Vortices of $\vec{E}$ and $\vec{H}$:

$$\nabla \times \vec{E} = - \frac{\partial B}{\partial t} \quad \text{— Faraday's law}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{j} \quad \text{— Ampere-Maxwell equ.}$$

b.) Sources of $\vec{D}$ and $\vec{B}$:

$$\nabla \cdot \vec{D} = \rho \quad \text{— Poisson equ. (Coulomb law)}$$

$$\nabla \cdot \vec{B} = 0$$

c.) Materials equations:

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_r \varepsilon_0 \vec{E} \qquad (\vec{P} = \varepsilon_0 \chi \vec{E},\ \chi: \text{tensor})$$

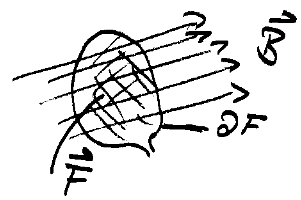
$$\vec{B} = \mu_0 (\vec{H} + \vec{M}) \qquad (\vec{M} = \psi \vec{H},\ \psi: \text{tensor})$$

$$\vec{j} = \sigma \cdot \vec{E} \qquad (\sigma_{ij} - \text{tensor!})$$

$\Rightarrow$

Next to the differential form of Maxwell's equation, we have the integral form of the equation, which are more general since they also describe the effects of timely-variable surface areas.

Faraday's law $$\boxed{\oint_{\partial F} \vec{E}\cdot d\vec{r} = -\frac{d}{dt}\int_F \vec{B}\cdot d\vec{A}}$$



Ampere-Maxwell $$\boxed{\oint_{\partial F} \vec{H}\cdot d\vec{r} = \frac{d}{dt}\int_F \vec{D}\cdot d\vec{A} + \int_F \vec{j}\cdot d\vec{A}}$$

Gauss's law $$\boxed{\oint_S \vec{D}\cdot d\vec{a} = Q} = \int_V \rho(\vec{r}\,')\,d^3\vec{r}\,'$$

Looking at the Maxwell's equation - note that they are not symmetric regarding $\vec{E}$ and $\vec{H}$.

The reason is that we have electrical monopoles - but <u>no</u> magnetic monopoles.

$\rightsquigarrow$ To the electrical charge density ρ - there exist <u>no analogue</u> magnetic charge density!

D.3 Vector potential

In electro-statics we had: $\nabla \times \vec{E} = 0$ - from which followed the existence of a scalar potential ϕ

$$\text{with} \quad \vec{E} = -\nabla\phi \quad \text{or} \quad D = -\varepsilon \nabla \phi$$

$\phi :=$ source potential

In magneto-static we introduced the vectorpotential $\vec{A}$.

If the vector potential $\vec{A} = A(\vec{r}, t)$ satisfies $\vec{B} = \nabla \times \vec{A}$, than the source equation

$$\boxed{\nabla \cdot \vec{B} = 0 = \nabla \cdot (\nabla \times \vec{A})} \qquad \vec{A} := \text{vortex potential}$$

is automatically fulfilled.

$\rightsquigarrow$ If the magnetic flux density $\vec{B}$ is given through the Vector potential $\vec{A}$, we do not need the equation $\nabla \cdot \vec{B} = 0$.

The Faraday's law of induction can be rewritten as

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt} = -\frac{d}{dt}(\nabla \times \vec{A})$$

$$\rightsquigarrow \quad \nabla \times \vec{E} + \nabla \times \frac{dA}{dt} = 0 \quad \text{or} \quad \boxed{\nabla \times (\vec{E} + \dot{\vec{A}}) = 0}$$

which means, for the function $(\vec{E} + \dot{\vec{A}})$ there exist a scalar Potential $\varphi(\vec{r}, t)$ with

$$\boxed{\vec{E} + \dot{\vec{A}} = -\nabla \varphi}$$

(see electro-statics $\nabla \times \vec{E} = 0$ ...) $\rightsquigarrow$

→ The magnetic flux density $\vec{B}$ and the electric field $\vec{E}$ can the expressed with help of the vector potential $\vec{A}$:

$$\boxed{\begin{aligned} \vec{B} &= \nabla\times\vec{A} \\ \text{and}\quad \vec{E} &= -\dot{\vec{A}} - \nabla\varphi \end{aligned}}$$

which automatically fulfill the equations $\underline{\nabla\cdot\vec{B}=0}$ and $\underline{\nabla\times(\vec{E}+\dot{\vec{B}})=0}$.

Problem:

For a given charge density $\rho(\vec{r})$ and current density $j(\vec{r})$ - find the associated fields $\vec{B}$ and $\vec{E}$. Assume a homogeneous and isotropic medium with ε_r = const. and μ_r = constant.

The fields can be found from the inhomogeneous DE

$$\nabla\cdot\vec{D} = \rho \qquad (\text{Poisson - Equ.})$$

$$\nabla\times\vec{H} = \frac{d\vec{D}}{dt} + \vec{j} \qquad (\text{Maxwell - Ampere})$$

Before we can determine the fields $\vec{E}$ and $\vec{B}$, we have to check whether the 4 potentials $\vec{A} = (A_x, A_y, A_z)$ and φ provide a unique solution of the fields.

(i) From 'Maxwell-Ampere' equ. follows

$$\nabla\times\vec{H} = \dot{\vec{D}} + \vec{j} \Rightarrow \mu(\nabla\times\vec{H}) = \nabla\times\vec{B} = \nabla\times(\nabla\times\vec{A})$$
$$= \mu(\dot{\vec{D}}+\vec{j}) \Rightarrow$$

$$\rightsquigarrow \quad \nabla(\nabla\cdot\vec{A}) - \Delta\vec{A} = \varepsilon\cdot\mu\,\dot{\vec{E}} + \mu\cdot\vec{j}$$

$$= -\varepsilon\mu\,\ddot{\vec{A}} - \varepsilon\mu\,\nabla\dot{\varphi} + \mu\cdot\vec{j}$$

Now use the relations $\frac{1}{v^2} := \varepsilon\cdot\mu = \varepsilon_r\cdot\mu_r\cdot\varepsilon_0\cdot\mu_0 = \frac{\varepsilon_r\cdot\mu_r}{c^2} = \frac{n^2}{c^2}$

with $n = \sqrt{\varepsilon_r\cdot\mu_r}$ = refractive index

to get:

$$\boxed{\left[\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \Delta\right]\vec{A} + \nabla\left(\nabla\cdot\vec{A} + \frac{1}{v^2}\dot{\varphi}\right) = \mu\cdot\vec{j}}$$

Maxwel - Ampere Equ. expressed in the potentials.

(ii) From Poisson equation follows

$$\nabla\cdot\vec{D} = \rho = \varepsilon\,\nabla\cdot\vec{E} = \varepsilon\,\nabla\cdot\left(-\dot{\vec{A}} - \nabla\varphi\right) = -\varepsilon\,\nabla\cdot\dot{\vec{A}} - \varepsilon\cdot\Delta\varphi$$

$$\rightsquigarrow \quad \boxed{\rho(\vec{r},t) = \varepsilon\cdot\left[\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \Delta\right]\varphi - \varepsilon\cdot\frac{\partial}{\partial t}\left(\nabla\cdot\vec{A} + \frac{1}{v^2}\dot{\varphi}\right)}$$

Since for the vector potential $\vec{A}$ only the vortices are fixed through $\nabla\times\vec{A} = \vec{B}$ but not the source $(\nabla\cdot\vec{A})$, we can freely choose $\boxed{\nabla\cdot\vec{A} = -\frac{1}{v^2}\dot{\varphi}}$ "Lorentz gauge"

$\hookrightarrow$ With this, the Maxwell - Ampere and Poisson equation $\Rightarrow$

simplify to the inhomogeneous wave equations

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\vec{A} = \mu \vec{j}$$ "Maxwell - Ampere"

and $$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\varphi = \frac{1}{\varepsilon}\rho(\vec{r},t)$$ "Poisson" Equation including Lorentz gauge

Since $\rho(\vec{r},t)$ and $\vec{j}(\vec{r},t)$ are given, $\vec{A}$ and φ can be found as a solution of the inhomogeneous DE.

Solution of the homogeneous DE can be added as long as they do not violate the "Lorentz-gauge".

In addition the choice of ρ and $\vec{j}$ have to satisfy the continuity equation $\boxed{\dot{\rho} + \nabla\cdot\vec{j} = 0}$

$$\dot{\rho} + \nabla\cdot\vec{j} = 0$$
$$= \left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\varepsilon\dot{\varphi} + \left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\frac{1}{\mu}\nabla\cdot\vec{A}$$
$$= \left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\cdot\frac{1}{\mu}\underbrace{\left(\nabla\cdot\vec{A} + \varepsilon\mu\dot{\varphi}\right)}_{=0}$$

$= 0$ through "Lorentz gauge" convention

D.4 Gauge invariance and gauge choice

The two homogeneous Maxwell equations:

$$\boxed{\nabla\cdot\vec{B}=0} \quad \text{and} \quad \boxed{\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}}$$

are automatically satisfied if we choose

$$\vec{B} = \nabla\times\vec{A}$$

$$\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t}$$

for arbitrary $\vec{A}$ and ϕ.

The Poisson equ. $\nabla\cdot E = \rho/\varepsilon_0$ and Maxwell-Ampere equ. $\nabla\times\vec{H} = \frac{dD}{dt}+\vec{j}$ become

$$\rho(r,t) = \varepsilon\left[\frac{1}{v^2}\frac{\partial^2}{\partial t^2}-\Delta\right]\phi - \varepsilon\frac{\partial}{\partial t}\left(\nabla\cdot\vec{A}+\frac{1}{v^2}\dot{\phi}\right)$$

$$\mu_0\vec{j} = \left[\frac{1}{v^2}\frac{\partial^2}{\partial t^2}-\Delta\right]\vec{A} + \nabla\left(\nabla\cdot\vec{A}+\frac{1}{v^2}\dot{\phi}\right)$$

a.) Lorentz gauge: $\nabla\cdot\vec{A}+\frac{1}{v^2}\frac{\partial\phi}{\partial t}=0$

b.) Coulomb gauge: $\nabla\cdot\vec{A}=0$

(a) Lorentz gauge is invariant for a gauge transformation

$$\vec{A}' = \vec{A}+\nabla\Lambda$$

$$\phi' = \phi - \frac{\partial\Lambda}{\partial t}$$

under the condition: $\left[\frac{1}{v^2}\frac{\partial^2}{\partial t^2}-\Delta\right]\Lambda = 0$

$\Rightarrow$

which means that Λ is a solution of the homogeneous wave equation.

Proof: put $\vec{A}'$ and ϕ' in Lorentz gauge convention!

Since Λ and with it $\partial\Lambda/\partial t$ and $\nabla\Lambda$ are solutions of the wave-equation, the modified potentials $\vec{A}'$ and ϕ' are again solutions of the inhomogeneous wave DE.

In addition, the Lorentz-gauge does not modify the fields $\vec{B}$ and $\vec{E}$:

<u>From</u> $\quad \vec{B} = \nabla \times \vec{A}$

follows $\quad \nabla \times (\vec{A} + \nabla\Lambda) = \nabla \times \vec{A} + \nabla \times \nabla\Lambda = \nabla \times \vec{A} = \vec{B}$

From $\quad \vec{E} = -\dot{\vec{A}} - \nabla\phi$

$$\leadsto \quad -\frac{\partial}{\partial t}\left(\vec{A} + \nabla\Lambda\right) - \nabla\left(\phi - \frac{\partial\Lambda}{\partial t}\right)$$

$$= -\dot{\vec{A}} - \underbrace{\frac{\partial}{\partial t}\nabla\Lambda}_{} - \nabla\phi + \underbrace{\nabla\left(\frac{\partial\Lambda}{\partial t}\right)}_{} = \underline{\underline{-\dot{\vec{A}} - \nabla\phi}}$$

(b) Coulomb-gauge is <u>not</u> Lorentz-invariant, but allows to decouple the wave equations and may be easier to solve than those in Lorentz gauge!

The four inhomogeneous wave-DE can be solved with help of Green's functions, which are defined by

$$\left[\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \Delta\right] G(\vec{r},t,\vec{r}',t') = 4\pi\,\delta(\vec{r}-\vec{r}')\,\delta(t-t')$$

The solution of the inhomg. DE's is than given by

$$\phi(\vec{r},t) = \int G(\vec{r},t,\vec{r}',t')\left[\frac{1}{\varepsilon}\rho(\vec{r}',t')\right] d^3\vec{r}'\,dt'$$

an

$$\vec{A}(\vec{r},t) = \int G(\vec{r},t,\vec{r}',t')\left[\mu\,\vec{j}(\vec{r}',t')\right] d^3\vec{r}'\,dt'$$

If there are boundary values for ϕ and $\vec{A}$, then they must be satisfied by the Green function!

For no specific boundary values, a retarded Green's function

$$\boxed{G(\vec{r},t,\vec{r}',t') = \frac{1}{4\pi}\,\frac{\delta\left[t'-t+\frac{1}{v}|\vec{r}-\vec{r}'|\right]}{|\vec{r}-\vec{r}'|}}$$

Can be used.

The time-dependance in the Green's function can be seperated

$$f(\vec{r},t) = h(\vec{r})\cdot g(t) \quad \ldots$$

Jackson 6.4 - 6.5

D5. Conservation of Energy and Momentum

1. Energy density and flow

Lets first look only at electromagnetic forces present on particles confined to a finite Volume. For speeds small compared to c, use Newton's equation

$$m_i \frac{d\vec{v}_i}{dt} = q_i \vec{E}(\vec{x}_i) + \vec{v}_i \times \vec{B}(x_i) \quad \Big| \cdot \vec{v}_i \text{ and } \sum_i$$

$$\rightsquigarrow \quad \sum_i \frac{d}{dt}\left(\frac{1}{2} m v_i^2\right) = \sum q_i \cdot \vec{v}_i \cdot \vec{E}$$

$$\text{or} \quad \frac{d}{dt} U_{kin} = \int_V \vec{j}(x) \cdot \vec{E} \, d^3\vec{x}$$

$$U_{kin} = \underbrace{\sum_i \frac{1}{2} m_i v_i^2}_{\text{total kinetic energy}} \qquad \vec{j}(x) = \sum_i q_i \vec{v}_i \, \delta(\vec{x} - \vec{x}_i) \quad \text{=: current density}$$

with defining the kinetic energy density

$$u_{kin}(\vec{x}) = \sum_i \frac{1}{2} m_i \vec{v}_i^{\,2} \delta(\vec{x} - \vec{x}_i)$$

$$\text{such that} \quad U_{kin} = \int_V u_{kin}(\vec{x}) \, d^3\vec{x}$$

we can write

$$\boxed{\int_V \left[\frac{\partial}{\partial t} u_{kin}(\vec{x}) - \vec{j}(\vec{x}) \cdot \vec{E}(\vec{x}) \right] d^3\vec{x} = 0}$$

$\Longrightarrow$

Another expression for "$\vec{j}(x) \cdot \vec{E}(x)$" can be found from Faraday-law and Ampere-law:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \Big| \cdot \vec{B}$$

$$\rightsquigarrow \quad \vec{B} \cdot (\nabla \times \vec{E}) + \vec{B} \cdot \left(\frac{\partial \vec{B}}{\partial t}\right) = 0 \qquad — \qquad (1)$$

and
$$\nabla \times \vec{B} = \frac{\partial D}{\partial t} + \mu \vec{j} \qquad \Big| \cdot \vec{E}$$

$$\rightsquigarrow \quad \vec{E} \cdot (\nabla \times \vec{B}) - \mu \varepsilon \, \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \mu \, \vec{j} \cdot \vec{E} \qquad (2)$$

$$(1)-(2): \; \underbrace{\vec{B} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{B})}_{\substack{= \nabla \cdot (\vec{E} \times \vec{B}) \\ = \mu \cdot (\vec{E} \times \vec{H})}} + \frac{\mu}{2} \cdot \frac{\partial}{\partial t}\left(\varepsilon \cdot E^2 + \frac{1}{\mu} B^2\right) = -\mu \, \vec{j} \cdot \vec{E}$$

$$\rightsquigarrow \quad \cancel{\mu} \nabla \cdot (\vec{E} \times \vec{H}) + \frac{\cancel{\mu}}{2} \frac{\partial}{\partial t}\left(\varepsilon E^2 + \frac{1}{\mu} \vec{B}^2\right) = -\cancel{\mu} \, \vec{j} \cdot \vec{E}$$

Define $\vec{S} := \vec{E} \times \vec{H} = \frac{1}{\mu} \vec{E} \times \vec{B}$:= Poynting vector

$[S] = \frac{VA}{m^2} = \frac{J}{m^2 \cdot s}$ = energy flow per area and time-unit

and $u_{em} = \frac{1}{2}\left(\varepsilon E^2 + \frac{1}{\mu} B^2\right)$ = electromagnetic energy density

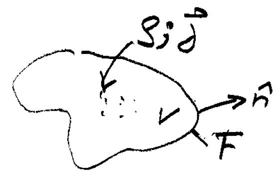
$$\rightsquigarrow \quad \nabla \cdot \vec{S} + \frac{\partial}{\partial t} u_{em} = \vec{j} \cdot \vec{E} \stackrel{!}{=} -\frac{\partial}{\partial t} u_{kin}$$

$$\rightsquigarrow \quad \boxed{\nabla \cdot \vec{S} + \frac{\partial}{\partial t}\left(u_{em} + u_{kin}\right) = 0}$$

Continuity equation for total energy density $u_{em} + u_{kin}$.

$\Longrightarrow$

$\rightsquigarrow$ Integrating over volume V bounded by the surface F



$$\int_V \nabla\cdot\vec{S}\, d^3\vec{r} + \frac{\partial}{\partial t}\Big[\underbrace{\int_V u_{em}\, d^3r}_{U_{em}} + \underbrace{\int_V u_{kin}\, d^3\vec{r}}_{U_{kin}}\Big] = 0$$

Gauss' theorem

$$= \oint_F \vec{S}\cdot\hat{n}\, d^2\vec{r} + \frac{\partial}{\partial t}(U_{em}+U_{kin}) = \boxed{0 = \oint_F \vec{S}\cdot\hat{n}\, d^2\vec{r} + \frac{\partial}{\partial t}U}$$

2. Conservation of Momentum

Total force on a collection of charged particles:

$$\vec{F} = \int d^3\vec{r}\, (\rho\cdot\vec{E} + \vec{j}\times\vec{B})$$

with Newton's 2^{nd}-Law, we have

$$\vec{F} = \frac{d}{dt}(m\cdot\vec{v}) = \frac{d\vec{P}_{mech}}{dt} = \int d^3\vec{r}\, (\underbrace{\rho}_{=\nabla\cdot\vec{D}}\cdot\vec{E} + \underbrace{\vec{j}}_{=\nabla\times\vec{H}-\dot{\vec{D}}}\times\vec{B})$$

$$\hookrightarrow \frac{d\vec{P}_m}{dt} = \int_V d^3r\, \Big[(\nabla\cdot\varepsilon\vec{E})\vec{E} + \Big(\frac{1}{\mu}\nabla\times\vec{B} - \varepsilon\frac{\partial\vec{E}}{\partial t}\Big)\times\vec{B}\Big]$$

$$= \int_V d^3r\cdot\Big[\varepsilon(\nabla\cdot\vec{E})\vec{E} + \frac{1}{\mu}\Big(\nabla\times\vec{B} - \varepsilon\mu\frac{\partial\vec{E}}{\partial t}\Big)\times\vec{B}\Big]$$

$$= \int_V d^3r\, \Big[\varepsilon(\nabla\cdot\vec{E})\vec{E} - \varepsilon\vec{E}\times(\nabla\times\vec{E}) + (\nabla\cdot\vec{B})\cdot\vec{B}/\mu - \vec{B}/\mu\times(\nabla\times\vec{B})\Big]$$

$$- \frac{d}{dt}\underbrace{\int_V d^3r\, (\vec{E}\times\vec{H})}_{=\text{ field momentum } = P_{field}}$$

$$\rightsquigarrow \boxed{\frac{d}{dt}(\vec{P}_m + \vec{P}_{field}) = \int_V d^3\vec{r}\, \frac{\partial T_{\alpha\beta}}{\partial T_\beta} = \oint_F d^2\vec{r}\, T_{\alpha\beta}\cdot\vec{n}_\beta}$$

(Tensor) momentum flux across surface

$$T_{\alpha\beta} = \varepsilon_0 E_\alpha E_\beta + \frac{1}{\mu_0} B_\alpha\cdot B_\beta - \frac{1}{2}\Big(\varepsilon_0 E^2 + \frac{1}{\mu}B^2\Big)\delta_{\alpha\beta}$$