Solutions for Homework # 6

Probem#1: Jackson 11.3: Show explicitly that two successive Lorentz transformations in the same direction are equivalent to a single Lorentz transformation with a velocity $v = \dfrac{v_1 + v_2}{1+(v_1 v_2 / c^2)}$ . .... *(20 Points)*

Let the move between frames Σ and Σ' be along x_1-direction. Then the Lorentz-transformation may be written in matrix form as

$$\begin{pmatrix} x_0' \\ x_1' \end{pmatrix} = \begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 \\ -\beta_1\gamma_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \qquad \left| \begin{array}{l} \beta_1 = v_1/c \\ \gamma_1 = \dfrac{1}{\sqrt{1-\beta_1^2}} \end{array} \right.$$

Similar, the transformation between frames Σ' and Σ'' is

$$\begin{pmatrix} x_0'' \\ x_1'' \end{pmatrix} = \begin{pmatrix} \gamma_2 & -\beta_2\gamma_2 \\ -\beta_2\gamma_2 & \gamma_2 \end{pmatrix} \begin{pmatrix} x_0' \\ x_1' \end{pmatrix}$$

The transformation between Σ and Σ'' is obtained by multiplying the matrices:

$$\begin{pmatrix} x_0'' \\ x_1'' \end{pmatrix} = \begin{pmatrix} \gamma_2 & -\beta_2\gamma_2 \\ -\beta_2\gamma_2 & \gamma_2 \end{pmatrix} \begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 \\ -\beta_1\gamma_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

$$= \begin{pmatrix} (1+\beta_1\beta_2)\gamma_1\gamma_2 & -(\beta_1+\beta_2)\gamma_1\gamma_2 \\ -(\beta_1+\beta_2)\gamma_1\gamma_2 & (1+\beta_1\beta_2)\gamma_1\gamma_2 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

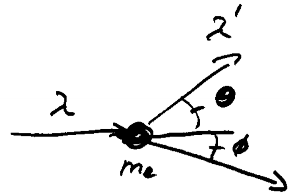
However, we want $\begin{pmatrix} x_0'' \\ x_1'' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\cdot\gamma & \gamma \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$

Compare: $\rightsquigarrow \gamma = \gamma_1\gamma_2(1+\beta_1\beta_2)$ and $\gamma\beta = \gamma_1\gamma_2\cdot(\beta_1+\beta_2)$

$$\rightsquigarrow \beta = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2} \quad \Rightarrow \quad \boxed{v = \frac{v_1+v_2}{1+v_1v_2/c^2}}$$

what had to be shown //

Problem #2: In a Compton effect, a γ-ray of wavelength λ strikes a free, but initially stationary, electron of mass m. The photon is scattered at an angle θ (measured from the incident direction), and its scattered wavelength is λ'. The electron recoils at an angle ϕ (measured from the incident direction). …..



a.)

$$\begin{rcases} (1) & E_{Ph} + E_e = E'_{Ph} + E'_e \\ (2) & P_{Ph} = P'_{Ph}\cdot\cos\theta + P'_e\cdot\cos\phi \\ (3) & 0 = P'_{Ph}\sin\theta + P'_e\sin\phi \end{rcases} \text{energy and momentum conservation}$$

Energy:

$$E_{Ph} = \frac{h\cdot c}{\lambda}, \quad E'_{Ph} = \frac{h\cdot c}{\lambda'}, \quad E_e = m_{e_0}c^2, \quad E'_e = m'c^2 = \gamma m_{e_0}c^2$$

momentums:

$$P_{Ph} = \frac{h}{\lambda}, \quad P'_{Ph} = \frac{h}{\lambda'}, \quad P_e'^2 = (m'^2 - m^2)c^2$$

b.) (2)^2+(3)^2: $P_{Ph}^2 + P_{Ph}'^2 - 2\cdot P'_{Ph}\cdot P_{Ph}\cdot\cos\theta = P_e'^2$

$$\hookrightarrow \frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2} - \frac{2\cdot h^2}{\lambda\cdot\lambda'}\cdot\cos\theta = (m'^2 - m^2)c^2 \qquad |(4)$$

from (1): $$\frac{h\cdot c}{\lambda} + m_e c^2 = \frac{hc}{\lambda'} + m'c^2 \qquad |(5)$$

(5)−(4): $$m_e^2\cdot c^2 + \frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2} + 2m_e\cdot c\cdot h\cdot\left[\frac{1}{\lambda} - \frac{1}{\lambda'}\right] - \frac{2h'}{\lambda\cdot\lambda'} - m_e^2c^2 - \frac{h^2}{\lambda^2}$$
$$- \frac{h^2}{\lambda'^2} + \frac{2h^2}{\lambda\cdot\lambda^2}\cdot\cos\theta = 0$$

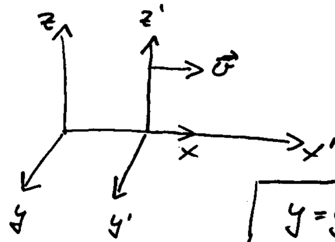
$$\rightsquigarrow 2m_e\cdot c\cdot h(\lambda' - \lambda) = 2h^2\cdot(1-\cos\theta) \rightsquigarrow \boxed{\lambda' - \lambda = \frac{2h\cdot\sin^2\theta/2}{m_e\cdot c}}$$

c.) $T = E - mc^2 = h\cdot\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)$, with $\lambda' - \lambda = (2h\sin^2\theta/2)/m_e c$

$$\rightsquigarrow T = h\cdot c\cdot\left[\frac{1}{\lambda} - \frac{1}{\lambda + \frac{2h\sin^2\theta/2}{m_e c}}\right] = \frac{h\cdot c}{\lambda}\cdot\frac{\left(\frac{2h}{\lambda\cdot mc}\right)\cdot\sin^2\theta/2}{1 + \frac{2h}{\lambda\cdot m\cdot c}\cdot\sin^2\theta/2} \quad //$$

Problem #3: Two events are specified in frame S by (5.0 m, 3.0 m, 0, 10 ns) and (8.0 m, 7.0 m, 0, 15 ns). Observer is in frame S', which moves in the x-direction at a constant speed relative to S, …..

$$P_1 = (5\,m,\ 3\,m,\ 0,\ 10\,ns)$$
$$P_2 = (8\,m,\ 7\,m,\ 0,\ 15\,ns)$$



$$\boxed{y = y',\ z = z'}$$

$$S_{12}^2 = -|\vec{x}_1 - \vec{x}_2|^2 + c^2(t_1 - t_2)^2$$

a.) $t_1' = \gamma(t_1 - \beta/c\, x_1)$, $t_2' = \gamma(t_2 - \beta/c\, x_2)$ with $t_1' = t_2'$

$$\rightsquigarrow \beta_x = c\cdot\frac{(t_1 - t_2)}{x_1 - x_2} = \frac{c \cdot 0.5\,ns}{3\,m} \approx 0.5\ ,\quad \gamma = \frac{1}{\sqrt{1-\beta^2}} = 1.154$$

$$v_x = 0.5 \cdot c$$

b.) $t' = \gamma(t - \beta/c\, x_1) \rightsquigarrow$

$$t_1' = 1.154\left(10\,ns - \frac{0.5}{c}\cdot 5\,m\right) \approx 2\,ns$$
$$t_2' = 1.154\left(15\,ns - \frac{0.5}{c}\cdot 8\,m\right) \approx 2\,ns \checkmark$$

c.) $x' = \gamma(x_1 - \beta\cdot c\cdot t_1)$, $y = y'$, $z = z'$

$$\Delta = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

in S : $\Delta = \sqrt{9 + 16} = 5\,m$

$$x_1' = 1.154\cdot(5 - 0.5\cdot c\cdot 10\,ns) \simeq 4.04$$
$$x_2' = 1.154\cdot(8 - 0.5\cdot c\cdot 15\,ns) \approx 6.64$$

in S' : $\Delta' = \sqrt{6.76 + 16} \simeq 4.77\,m$ or $\Delta = \gamma\cdot\Delta'$

d.) $x_1' = x_2'$; $\gamma(x_1 - \beta c\cdot t_1) = \gamma(x_2 - \beta\cdot c\cdot t_2)$; $\beta = \frac{1}{c}\,\frac{(x_1 - x_2)}{t_1 - t_2} = \frac{(8-5)\,m}{(15-10)\,ns}\cdot\frac{1}{c}$

$\rightsquigarrow \beta \simeq 2$, $v_x \simeq 2c > c \Rightarrow$ it is impossible

e.) $S_{1,2}^2 = c^2(t_1 - t_2)^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2 - (z_1 - z_2)^2 = c^2\cdot(5\,ns)^2 - (3\,m)^2 - (4\,m)^2$

$= 9\cdot 10^{16}\cdot 25\cdot 10^{-18} - 9 - 16 \approx -22.5 \underline{< 0} \rightsquigarrow$ it is a space-Line separation! //

Problem #4: Find the fields of a uniformly moving charged particle by the following steps …. *(20 Points)*

a) $\varphi = \frac{q}{r'}$, $A = 0$, $A' = (\varphi', 0, 0, 0)$

b.) $A = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} A'$,

$\varphi = \gamma \varphi'$

$A_1 = \beta \gamma \varphi'$

$A_2 = 0$

$A_3 = 0$

φ' in K system: $\frac{q}{r} = \frac{q}{\sqrt{(x - v \cdot t)^2 \cdot \gamma + y^2 + z^2}}$

c.) $E = -\frac{1}{c}\frac{\partial A}{\partial t} - \nabla \varphi$, $\vec{B} = \vec{\nabla} \times \vec{A}$

$$\vec{E} = -\left[\frac{\beta \gamma q}{c} \cdot \frac{(x - vt) \cdot v \gamma^2}{[(x-vt)^2\gamma^2 + y^2 + z^2]^{3/2}} \cdot \hat{e}_x - \frac{\gamma^3 q \, (x - v \cdot t)}{[(x-vt)^2\gamma^2 + y^2 + z^2]^{3/2}} \cdot \hat{e}_x \right.$$

$$\left. - \hat{e}_y \underbrace{\frac{\gamma \cdot q \cdot y}{[(x-vt)^2\gamma^2 + y^2 + z^2]^{3/2}}}_{= E_y} - \hat{e}_z \underbrace{\frac{\gamma \cdot q \cdot z}{[(x-vt)^2\gamma^2 + y^2 + z^2]^{3/2}}}_{= E_z} \right]$$

$$= + \left[\hat{e}_x \cdot \frac{q\gamma(x - vt)}{((x-vt)^2\gamma^2 + y^2 + z^2)^{3/2}} + \hat{e}_y \frac{\gamma q \cdot y}{((x-vt)^2\gamma^2 + y^2 + z^2)^{3/2}} + \hat{e}_z \frac{\gamma q \cdot z}{((x-vt)^2\gamma^2 + y^2 + z^2)^{3/2}} \right]$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A & 0 & 0 \end{vmatrix} = \frac{\partial A_1}{\partial z} \cdot \hat{j} - \frac{\partial A_1}{\partial y} \hat{k}$$

$$= \frac{\gamma \beta \cdot q}{((x-vt)^2\gamma^2 + y^2 + z^2)^{3/2}} \cdot (-z \cdot \hat{j} + y \hat{k})$$

Problem #5: Jackson 11.13: An infinitely long straight wire of negligible cross-sectional area is at rest and has a uniform linear charge density q_0 in the inertial frame K'. ….. *(20 Points)*

a.) $\oint_S \vec{E}\, d^2r = \frac{Q_{encl.}}{\varepsilon_0} = h E_r \, 2\pi r = h\, q_0/\varepsilon_0 \rightsquigarrow E_r = \frac{q_0}{2\pi\varepsilon_0 r}$

$E_\theta = 0$ and $E_z = 0$ by Gaus law and $\boxed{\vec{B} = 0}$ since there is no current in this frame.

From Jackson eq. 11.149: $E' = \gamma(\vec{E} + \vec{\beta}\times\vec{B}) - \frac{\gamma^2}{\gamma+1}\cdot\vec{\beta}(\vec{\beta}\cdot\vec{E})$

$$B' = \gamma(\vec{B} - \vec{\beta}\times\vec{E}) - \frac{\gamma^2}{\gamma+1}\cdot\vec{\beta}(\vec{\beta}\cdot\vec{B})$$

$\rightsquigarrow E_r' = \gamma\,\hat{r}\cdot|\vec{E}| = \hat{r}\cdot\frac{\gamma q_0}{2\pi\varepsilon_0 r}$, $E_\theta' = 0 = E_z'$

and $B_\theta' = \gamma(-\beta E_r) = -\beta\gamma\frac{q_0}{2\pi\varepsilon_0 r}$, $B_r' = 0$, $B_z' = 0$

[In SI - units: $E_r' = \frac{\gamma q_0}{2\pi\varepsilon_0 r}$, $B_\theta' = -\frac{1}{c}\beta\cdot\gamma\frac{q_0}{2\pi\varepsilon_0 r}$]

b.) In K-system we have: $j^\mu = (c\cdot\rho, j_x, j_y, j_z)$

$$= \left(\frac{c\cdot q_0\,\delta(r)}{2\pi r}, 0, 0, 0\right)$$

$$L = \begin{pmatrix} \gamma & -\beta\cdot\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\rightsquigarrow$ To transform in K'-frame: $j^{\mu\prime} = L\cdot j^\mu$

$$\rightsquigarrow j^{\mu\prime} = \ldots = \left(\frac{c\cdot\gamma q_0}{2\pi r}\cdot\delta(r), -\frac{c\cdot\gamma\cdot\beta\cdot q_0}{2\pi r}\cdot\delta(r), 0, 0\right)$$

c.) For K'-frame, we found $j^{\mu\prime}$, charge density ρ in K-frame transforms to ρ' in K'-frame: $\boxed{\rho' = \gamma\rho}$; $\vec{j}' = -\beta\gamma\cdot c\cdot q_0\,\delta(r)\cdot\hat{z} = -\frac{\beta\gamma q_0}{c\cdot\mu_0\varepsilon_0}\delta(r)\cdot\hat{z}$

$\hookrightarrow \boxed{E_r' = \frac{\lambda'}{2\pi\varepsilon_0 r} = \frac{\gamma\cdot q_0}{2\pi\varepsilon_0 r}}$ and $B_\theta' = \frac{\mu_0 J_z'}{2\pi r} = \frac{\mu_0}{2\pi r}\cdot\frac{(-\beta\gamma q_0)}{c\cdot\mu_0\varepsilon_0} = -\frac{1}{c}\beta\cdot\gamma\cdot\frac{q_0}{2\pi\varepsilon_0 r}$ //