Solutions for Homework # 5

Probem#1: Jackson Problem 8.2 (a), (b), and (c), textbook (*30 Points)*

a.) $E_{TEM} = -\nabla_t \varphi \;;\qquad \Delta_t \varphi = 0$

$\rightsquigarrow$ cylindrical coord. (ρ, φ, z): $\quad \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\varphi\right) = 0$

$$\rightsquigarrow \partial\varphi = C \cdot \frac{\partial \rho}{\rho} \quad \Rightarrow \quad \varphi = C \cdot \ln(\rho)$$

$$\hookrightarrow E_{TEM} = -\frac{C}{\rho} \cdot \hat{e}_r \cdot e^{ik\cdot z - i\omega t}$$

$$H_{TEM} = \sqrt{\frac{\varepsilon}{\mu}} \cdot \hat{e}_z \times \vec{E}_{TEM} = -\sqrt{\frac{\varepsilon}{\mu}} \cdot \frac{C}{\rho} \cdot \hat{e}_\varphi \cdot e^{ik\cdot z - i\omega t}$$

boundary condition: $\quad H_{TEM}\big|_{\rho = a} = H_0 \;\rightsquigarrow\; C = -\sqrt{\frac{\mu}{\varepsilon}} \cdot a \cdot H_0$

$$\hookrightarrow \vec{H}_{TEM} = -H_0 \frac{a}{\rho}\,\hat{e}_\varphi\, e^{ik\cdot z - i\omega t} \;;\quad \vec{E}_{TEM} = H_0 \cdot \frac{a}{\rho} \cdot \sqrt{\frac{\mu}{\varepsilon}} \cdot \hat{e}_r \cdot e^{ik\cdot z - i\omega t}$$

$$P = \int \langle \vec{S} \rangle\, d\vec{a} = \frac{1}{2} \cdot \int_a^b \mathrm{Re}\left\{ \int_0^{2\pi} (\vec{E} \times \vec{H}^*)\, \rho\, d\rho\, d\varphi \right.$$

$$= \pi \sqrt{\frac{\mu}{\varepsilon}}\, a^2 H_0^2 \int_a^b \frac{1}{\rho}\, d\rho$$

$$= \underline{\underline{\sqrt{\frac{\mu}{\varepsilon}}\, \pi a^2 H_0^2 \ln\left(\frac{b}{a}\right)}}$$

b.) see Jackson page 364, equ. 8.56 - 58:

attenuation constant $\gamma = -\frac{1}{2P}\frac{\partial P}{\partial z} \;;\quad -\frac{\partial P}{\partial z} = \frac{1}{2\sigma\delta} \oint |\hat{n} \times \vec{H}|^2\, dl$

$$\hookrightarrow \frac{dP}{dz} = \frac{1}{2\sigma\delta}\, a^2 \cdot H_0^2\, 2\pi \left[\frac{1}{a} + \frac{1}{b}\right] \;,\qquad P = P_0 \cdot e^{-2\gamma \cdot z} \qquad \Rightarrow$$

Problem #1 continued:

$$\gamma = -\frac{1}{2P}\frac{dP}{dz} = \frac{\frac{1}{2\sigma\delta} a^2 H_0^2 \cdot 2\pi\left(\frac{1}{a}+\frac{1}{b}\right)}{2\cdot\sqrt{\frac{\mu}{\varepsilon}}\,\pi a^2 H_0^2 \ln(b/a)} = \frac{1}{2\sigma\delta}\sqrt{\frac{\varepsilon}{\mu}}\cdot\frac{\left(\frac{1}{a}+\frac{1}{b}\right)}{\ln(b/a)}$$

8.2 (c.)

$$\oint \vec{H}_{TEM}\, dl = J = H_0\cdot a\int_0^{2\pi}\frac{\hat{e}_\varphi}{\rho}\cdot\rho\, d\varphi$$

$$= 2\pi H_0\cdot a$$

$$J = 2\pi H_0\cdot a\; e^{ikz-i\omega t}$$

Voltage $V = \int_a^b \vec{E}\, d\rho = \int_a^b \sqrt{\frac{\mu}{\varepsilon}}\cdot a\cdot H_0\,\frac{1}{\rho}\, d\rho$

$$= \sqrt{\mu/\varepsilon}\cdot a\cdot H_0\cdot\ln(b/a)$$

$$\rightsquigarrow V = \sqrt{\frac{\mu}{\varepsilon}}\cdot a\cdot H_0 \ln\left(\frac{b}{a}\right)\cdot e^{ikz-i\omega t}$$

Impedance $Z_0 = \frac{V}{J} = \frac{\sqrt{\mu/\varepsilon}\; a\; H_0 \ln(b/a)\; e^{ikz-i\omega t}}{2\pi H_0\cdot a\; e^{i(kz-\omega t)}}$

$$= \frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}}\ln\left(\frac{b}{a}\right)$$

Problem #2: In a rectangular waveguide, assuming a=b

a. Find the explicit fields for TE_{21} mode.
b. Find the cutoff frequency ω_{12} and compare with ω_{10}.
c. Find the total average transmitted power (Jackson p. 363…).. *(30 Points)*

a.) TE_{21} - mode ?

$$E_z = 0, \quad H_z = \underbrace{H_0 \cos\left(\frac{2\pi}{a}\cdot x\right)\cdot \cos\left(\frac{\pi}{b}\right)\cdot}_{=\psi(x,y)} e^{ik\cdot z - i\omega t}$$

$$|H_t| = \bar{H}_t = \pm \frac{ik}{\gamma^2} \nabla_t \psi(x,y) \qquad \underline{\text{with:}}\ \gamma_{mn}^2 = \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]$$

$$\gamma^2 = \gamma_{21}^2 = \pi^2\left[\frac{4}{a^2} + \frac{1}{b^2}\right] = \frac{5\pi^2}{a^2} \quad (a=b)$$

$$\hookrightarrow H_x = +\frac{ik}{\gamma^2}\cdot(-1)\cdot H_0 \cdot \sin\left(\frac{2\pi}{a}x\right)\cdot \cos\left(\frac{\pi}{b}\cdot y\right)\cdot \frac{2\pi}{a}\cdot e^{ik\cdot z - i\omega t}$$

$$H_y = -\frac{ik\cdot\pi}{\gamma^2}\cdot H_0 \cos\left(\frac{2\pi}{a}\cdot x\right)\cdot \sin\left(\frac{\pi}{b}\cdot y\right)\cdot e^{ik\cdot z - i\omega t}$$

$$\vec{H}_t = +\frac{k}{\omega\cdot\mu}\hat{z}\times\vec{E}_t \quad \rightsquigarrow \quad \vec{E}_t = -\frac{\omega\cdot\mu}{k}\hat{z}\times\vec{H}_t \Rightarrow E_z = 0 \checkmark$$

$$E_x = \frac{\omega\cdot\mu}{k} H_y = -\frac{i\pi\mu\omega}{a\cdot\gamma^2} H_0 \cos\left(\frac{2\pi}{a}\cdot x\right)\cdot \sin\left(\frac{\pi}{b}\cdot y\right)\cdot e^{ik\cdot z - i\omega t}$$

$$E_y = -\frac{\omega\cdot\mu}{k}\cdot H_x = +\frac{i\omega\cdot\mu\cdot 2\pi}{b\cdot\gamma^2}\cdot H_0 \sin\left(\frac{2\pi}{a}\cdot x\right)\cdot\cos\left(\frac{\pi}{b}\cdot y\right)\cdot e^{ik\cdot z - i\omega t}$$

b.) $$\omega_{21} = \frac{\pi}{\sqrt{\mu\cdot\varepsilon}}\left[\left(\frac{2}{a}\right)^2 + \left(\frac{1}{b}\right)^2\right]^{1/2}, \text{ for } a=b \rightsquigarrow \omega_{21} = \frac{\pi}{\sqrt{\mu\varepsilon}}\cdot\frac{\sqrt{5}}{a} = \frac{\pi}{a}\cdot\sqrt{\frac{5}{\mu\varepsilon}}$$

$$\omega_{10} = \frac{\pi}{\sqrt{\mu\varepsilon}}\cdot\frac{1}{a} \quad \rightsquigarrow \quad \omega_{21}/\omega_{10} = \sqrt{5}$$

c.) $$P = \int \langle S\rangle\, da = \int_0^a\int_0^{b=a} \frac{1}{2}\mathrm{Re}\left[\vec{E}\times\vec{H}^*\right]\cdot\hat{e}_z\, dx\,dy = \frac{1}{2}\int\left[\varepsilon\,\bar{E}\cdot\bar{E}^* + \mu\,\bar{H}\cdot\bar{H}^*\right]\cdot\hat{e}_z\, da$$

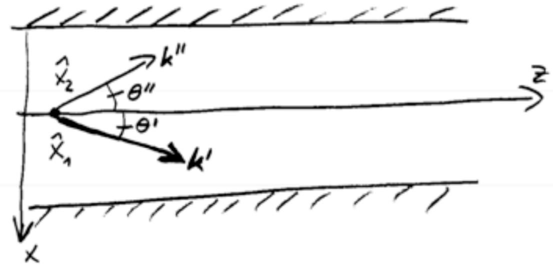
$$= \frac{1}{2}\int \mathrm{Re}\left(E_x\cdot H_y^* - E_y\cdot H_x^*\right)da = \frac{1}{2}\int\left[\frac{k a^2\mu\omega}{25\pi^2}H_0^2\cos^2\left(\frac{2\pi}{a}\cdot x\right)\sin^2\left(\frac{\pi}{a}\cdot y\right) + \frac{4ka^2\mu\omega}{25\pi^2}H_0^2\sin^2\left(\frac{2\pi x}{a}\right)\cos^2\left(\frac{\pi}{a}y\right)\right]$$

$$= \int_0^a\int_0^{b=a}\left[\frac{ka^2\mu\omega}{200\pi^2}H_0\cdot\left\{\cos^2\left(\frac{2\pi}{a}x\right)\cdot\sin^2\left(\frac{\pi}{a}\cdot y\right) + 4\sin^2\left(\frac{2\pi}{a}x\right)\cos^2\left(\frac{\pi}{a}y\right)\right\}dx\,dy = \frac{k\cdot a^4\cdot\mu\cdot\omega}{40\pi^2}H_0^2\right.$$

Problem #3: Equation 8.46 (Jackson p. 362) shows that the TE_{10} field is equivalent to two plane waves traveling in oblique directions.

a. Find the phase velocity of the TE_{10} wave.
b. Find the phase velocity of the two equivalent plane waves.
c. Find the group velocity of the TE_{10} wave. *(20 Points)*

$$E_y = i\omega\mu \frac{a}{\pi} H_0 \sin\left(\frac{\pi}{a}x\right) e^{ik\cdot z - i\omega t}$$



use:

$$i\cdot \sin\alpha = (e^{i\alpha} - e^{-i\alpha})/2$$

$$v_{phase} = \frac{dx}{dt}; \quad k\cdot z - \omega t = \text{const.} \quad = \omega/k$$

$$\hookrightarrow E_y = \underbrace{\frac{\omega\mu\cdot a}{2\pi} H_0}_{=E_0} \cdot \left[e^{i\frac{\pi}{a}x + i\cdot k\cdot z - i\omega t} - e^{i(-\frac{\pi}{a}\cdot x + k\cdot z - \omega\cdot t)} \right]$$

$$= E_0 \cdot e^{i(k'\cdot \bar{x} - \omega t)} + E_0\, e^{i(k''\cdot \bar{x} - \omega t)}$$

$$\Rightarrow \bar{k}'\cdot\bar{x} = k'\cdot\cos\theta'\cdot z + k'\cdot\sin\theta'\cdot x = \frac{\pi}{a}x - k\cdot z;$$

$$\bar{k}' = \sqrt{\left(\frac{\pi}{a}\right)^2 + k^2}\cdot \hat{x}_1$$

$\left[\begin{array}{l} \bar{k} = |k| \\ \bar{x} = |x| \\ \text{absolute value} \end{array}\right]$

$$\bar{k}''\cdot\bar{x} = k''\cdot\cos\theta''\cdot z + k''\cdot\sin\theta''\cdot x = -\frac{\pi}{a}\cdot x - k\cdot z;$$

$$\bar{k}'' = \sqrt{\left(\frac{\pi}{a}\right)^2 + \bar{k}^2}\cdot \hat{x}_2$$

$$\hookrightarrow E_{x_1} = E_0\, e^{i(|k'|\cdot x_1 - \omega t)}; \quad E_{x_2} = E_0\cdot e^{i(|k''|\cdot x_2 - \omega\cdot t)}$$

$$E_x = E_{x_1} + E_{x_2}, \quad \omega = \frac{\sqrt{(\pi/a)^2 + k^2}}{\sqrt{\mu\cdot\varepsilon}} \quad \left(\begin{array}{l} \text{use } k^2 = \omega^2\mu\varepsilon - \left(\frac{\pi}{a}\right)^2 \\ \rightsquigarrow \omega = \ldots \end{array} \right)$$

b) $$v_{ph,x_1} = \frac{dx_1}{dt}\Big|_{\bar{k}_1 x - \omega t = \text{const}} = \frac{\omega}{\sqrt{k^2 + (\pi/a)^2}} = \frac{1}{\sqrt{\mu\varepsilon}}$$

c.) $$v_g = \frac{d\omega}{dk} = \frac{1}{\sqrt{\mu\varepsilon}} \frac{k}{\sqrt{k^2 + (\pi/a)^2}} = \frac{k}{\mu\varepsilon\cdot\omega}$$

Problem #4: In a rectangular resonant cavity,

a. Find the explicit fields for TE_{111} mode.
b. Find the resonant frequency ω_{111} by assuming a=b=d.
c. Find the time-averaged energy stored in the cavity. *(20 Points)*

a.) TEM_{111} ? $E_z = 0$, $H_z = \underbrace{H_0 \cos\left(\frac{\pi}{a}x\right)\cdot\cos\left(\frac{\pi}{b}\cdot y\right)}_{=\psi(x,y)}\cdot\sin\left(\frac{\pi}{d}\cdot z\right)\cdot e^{-i\omega t}$

$$\gamma^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$

$$|H_t| = \bar{H}_t = \frac{\pi}{d\gamma^2}\cos\left(\frac{\pi}{d}\cdot z\right)\cdot\nabla_t\psi(x,y)$$

$$H_x = -\frac{\pi^2}{d\cdot\gamma^2\cdot a}\cdot H_0\sin\left(\frac{\pi}{a}x\right)\cdot\cos\left(\frac{\pi}{b}\cdot y\right)\cdot\sin\left(\frac{\pi}{d}\cdot z\right)e^{-i\omega t}$$

$$H_y = -\frac{\pi^2}{\gamma^2 d\cdot b}\cdot H_0\cos\left(\frac{\pi}{a}x\right)\cdot\sin\left(\frac{\pi}{b}\cdot y\right)\cdot\sin\left(\frac{\pi}{d}\cdot z\right)\cdot e^{-i\omega t}$$

$$\left[\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 0 & 1\\ \psi_x' & \psi_y' & 0\end{vmatrix}\right]$$

$$|E_t| = \bar{E}_t = -\frac{i\omega\mu}{\gamma^2}\sin\left(\frac{\pi}{d}\cdot z\right)\cdot\hat{z}\times\nabla_t\psi(x,t)$$

$$\hookrightarrow E_x = -\frac{i\omega\mu\pi}{\gamma^2 b}\sin\left(\frac{\pi}{d}\cdot z\right)\cdot\cos\left(\frac{\pi}{a}\cdot x\right)\cdot\sin\left(\frac{\pi}{b}\cdot y\right)\cdot H_0\cdot e^{-i\omega t}$$

$$E_y = \frac{i\omega\mu}{\gamma^2}\sin\left(\frac{\pi}{d}z\right)\cdot\psi'_{x(x,y)} = -\frac{i\omega\mu\cdot\pi}{\gamma^2\cdot a}\sin\left(\frac{\pi}{d}\cdot z\right)\cdot\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{b}\cdot y\right)\cdot H_0 e^{-i\omega t}$$

$$E_z = 0$$

b) $\omega_{111} = ?$, $a = b = d$

$$\gamma^2 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2 = 2\left(\frac{\pi}{a}\right)^2,\quad \omega_{111}^2 = \frac{1}{\mu\varepsilon}\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2 + \left(\frac{\pi}{d}\right)^2\right] = \frac{3}{\mu\varepsilon}\left(\frac{\pi}{a}\right)^2 \rightsquigarrow \underline{\underline{\omega_{111} = \frac{\pi}{a}\sqrt{\frac{3}{\mu\varepsilon}}}}$$

c.)
$$\langle u\rangle = \frac{1}{4}\int_0^a\int_0^b\int_0^d \mathrm{Re}\left[\varepsilon\,\bar{E}\cdot\bar{E}^* + \mu\,\bar{H}\cdot\bar{H}^*\right]dx\,dy\,dz$$

$$= \frac{1}{4}\int\left[\mu\left(H_z^2 + H_y^2 + H_x^2\right) + \varepsilon\left(E_x^2 + E_y^2\right)\right]dx\,dy\,dz$$

$$= \frac{1}{4}H_0^2\left[\frac{\mu\cdot a\cdot b\cdot d}{8} + \frac{\mu\cdot a\cdot b\cdot d}{\left(1/a^2 + 1/b^2\right)^2\cdot 8} + \frac{\mu\cdot a\cdot b\cdot d}{8d^2\cdot b^2\cdot\left(1/a^2 + 1/b^2\right)^2} + \frac{\mu\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{d^2}\right)a\cdot b\cdot d}{8\cdot b^2\left(1/a^2 + 1/b^2\right)^2}\right.$$

$$\left. + \frac{\mu\cdot\left(1/a^2 + 1/b^2 + 1/d^2\right)a\cdot b\cdot d}{8\left(1/a^2 + 1/b^2\right)^2\cdot a^2}\right] = \frac{\mu H_0^2\cdot a\cdot b\cdot d}{16}\cdot\frac{\left(1/a^2 + 1/b^2 + 1/d^2\right)}{1/a^2 + 1/b^2}$$

for $a = b = d$: $\langle u\rangle = 3/32\cdot\mu\cdot H_0^2 a^3$