Solutions for Homework # 4

1. The electric field for an elliptically polarized plane wave is $\mathbf{E}=\mathbf{E}_1+\mathbf{E}_2$, where $E_1 = \hat{e}_x \cdot E_1\, e^{i\cdot(k\cdot z-\omega\cdot t+\alpha)}$ and $E_2 = \hat{e}_y \cdot E_2\, e^{i\cdot(k\cdot z-\omega\cdot t+\beta)}$. Calculate the average energy flow for such a wave. (15points)
   (a) Does the energy flow depend on the phases α and β? Assume that E_1 and E_2 are real quantities.
   (b) Determine the polarization state of $\mathbf{E}=\mathbf{E}_1+\mathbf{E}_2$!

a.) $\vec{E} = \vec{E}_1 + \vec{E}_2$, [for plane wave : $\vec{H} = \sqrt{\frac{\varepsilon}{\mu}} \cdot \hat{n} \times \vec{E}$]

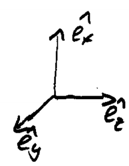
$\vec{E}_1 = \hat{e}_x\, E_1 \cdot e^{i(kz-\omega t+\alpha)} \rightsquigarrow \vec{H}_1 = \hat{e}_y \sqrt{\frac{\varepsilon}{\mu}} \cdot E_1 \cdot e^{i(k\cdot z-\omega t+\alpha)}$

$\vec{E}_2 = \hat{e}_y\, E_2\, e^{i(kz-\omega t+\beta)} \rightsquigarrow \vec{H}_2 = -\hat{e}_x \sqrt{\frac{\varepsilon}{\mu}}\, E_2 \cdot e^{i(k\cdot z-\omega t+\beta)}$

$$\langle \vec{S} \rangle = \frac{1}{2} \mathrm{Re}(\vec{E} \times \vec{H}^*) = \frac{1}{2}\left[(\vec{E}_1 + \vec{E}_2) \times (\vec{H}_1 + \vec{H}_2) \right]$$

$$= \frac{1}{2} \mathrm{Re}\left\{ \vec{E}_1 \times \vec{H}_1^* + \underbrace{\vec{E}_1 \times \vec{H}_2^*}_{=0} + \underbrace{\vec{E}_2 \times H_1^*}_{=0} + \vec{E}_2 \times \vec{H}_2^* \right\}$$

since $\hat{e}_y \times \hat{e}_y = 0$, $\hat{e}_x \times \hat{e}_x = 0$



$$= \frac{1}{2} \mathrm{Re}\left\{ \hat{e}_x \times \hat{e}_y \cdot E_1 \cdot e^{i(kz-\omega t+\alpha)} \cdot \sqrt{\varepsilon/\mu} \cdot E_1 \cdot e^{-i(kz-\omega t+\alpha)} \right.$$
$$\left. + \hat{e}_y \times (-\hat{e}_x) \cdot E_2 \sqrt{\varepsilon/\mu}\, E_2 \cdot e^{i(kz-\omega t+\beta)} \cdot e^{-i(kz-\omega t+\beta)} \right\}$$

$$= \frac{1}{2} \mathrm{Re}\left\{ \hat{e}_z\, E_1^2 \sqrt{\varepsilon/\mu} + \hat{e}_z\, E_2^2 \cdot \sqrt{\frac{\varepsilon}{\mu}} \right\}$$

$$= \frac{1}{2} (E_1^2 + E_2^2) \cdot \sqrt{\frac{\varepsilon}{\mu}} \cdot \hat{e}_z$$

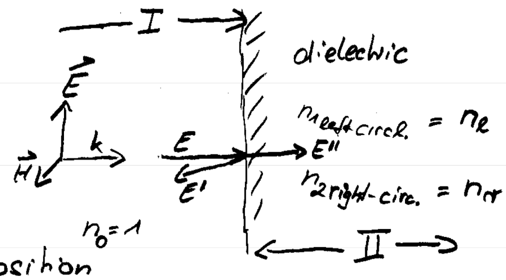
$\rightsquigarrow$ energy flow does not depend on phase.

b.) E_1, E_2 are complex amplitudes $\rightsquigarrow E_1 = E_{0x}\, e^{i\varphi_1}$; add phase $\alpha \rightsquigarrow E_{0x} \cdot e^{i\varphi_1 + \alpha}$
$E_2 = E_{0y}\, e^{i\varphi_2}$; " " $\beta \rightsquigarrow E_{0y}\, e^{i\varphi_2 + \beta}$
E_{0x}, E_{0y} - real

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \underbrace{\left[\hat{e}_x \cdot E_{0x} \cdot e^{i(\varphi_1+\alpha)} + \hat{e}_y\, E_{0y}\, e^{i(\varphi_2+\beta)} \right]}_{=\text{elliptical polarized}} \cdot e^{i(k\cdot z-\omega t)}$$

if not further specified!

Problem #2: A linearly polarized wave $E_x = E_o\, e^{i(k\cdot z - \omega\cdot t)}$ is normally incident onto a dielectric medium. The medium has indices of refraction n_1 and n_2 for left-circularly and right-circularly polarized light, respectively. Find the reflection coefficient R. *(15 Points)*



$$E_x = E_0\, e^{i(k\cdot z - \omega t)}$$

linearly polarized light is a superposition of two circularly polarized waves

$$\vec{E} = \frac{1}{2} E_0 (\hat{e}_x + \hat{e}_y)\, e^{i(k\cdot z - \omega t)} + \frac{1}{2} E_0 (\hat{e}_x - i\hat{e}_y)\cdot e^{i(k\cdot z - \omega t)}$$

$$= \vec{E}_\ell + \vec{E}_r$$

Boundary condition on interface: $E_I\big|_\perp = E_{II}\big|_\perp$, $H_I\big|_\perp = H_{II}\big|_\perp$

left circular:

$$\rightsquigarrow E_\ell + E_\ell' = E_\ell''$$

$$\sqrt{\frac{\varepsilon_0}{\mu_0}}\cdot(E_\ell - E_\ell') = \sqrt{\frac{\varepsilon_\ell}{\mu_\ell}}\cdot E_\ell''$$

right circular:

$$E_r + E_r' = E_r''$$

$$\sqrt{\frac{\varepsilon_0}{\mu_0}}\cdot(E_r - E_r') = \sqrt{\frac{\varepsilon_r}{\mu_r}}\cdot E_r''$$

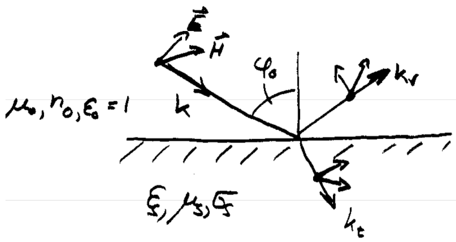
$$\rightsquigarrow E_\ell' = \frac{n_0 - n_\ell}{n_0 + n_\ell}\cdot E_\ell, \quad E_r' = \frac{n_0 - n_r}{n_0 + n_r}\cdot E_r, \quad \mu_0 = \mu_\ell = \mu_r = 1$$

$$R = \left(\frac{E_\ell'}{E_\ell}\right)\left(\frac{E_\ell'}{E_\ell}\right)^* + \left(\frac{E_r'}{E_r}\right)\cdot\left(\frac{E_r'}{E_r}\right)^* = \frac{1}{2}\left\{\left(\frac{n_0 - n_\ell}{n_0 + n_\ell}\right)^2 + \left(\frac{n_0 - n_r}{n_0 + n_r}\right)^2\right\}$$

↑ since we split wave in two!

Problem #3: Problem 7.4, Jackson textbook (30 points)

plane wave : $\vec{E}_i = E_{0i}\, e^{i(k\cdot z - \omega t)}$, $\vec{H}_i = \sqrt{\frac{\varepsilon_0}{\mu_0}}\; \hat{n} \times \vec{E}_i$



$\varphi_0 = 0$: normal incidence

$\rightsquigarrow\ r_p = r_s = \dfrac{|E_{0r}|}{|E_{0i}|}$, $R = r_s \cdot r_s^*$

reflected wave : $\vec{E}_r = E_{0r}\, e^{-i(k\cdot z + \omega t)}$

$\vec{H}_r = \sqrt{\frac{\varepsilon_0}{\mu_0}}\, e^{-i(k\cdot z + \omega t)} \cdot [-\hat{n} \times \vec{E}_r]$

transmitted wave : $\vec{E}_t = E_{0t}\, e^{i(k'\cdot z - \omega t)}$

$\vec{H}_t = \dfrac{1}{\omega\mu}\, k' \cdot \hat{n} \times \vec{E}_t$, $k' = \omega \cdot \sqrt{\varepsilon_s \mu_s} = \dfrac{\omega}{v}$

Maxwell boundary conditions for perpendicular ($\perp$) field components

$$E_i\big|_\perp = E_r\big|_\perp \quad \text{or} \quad [E_{0i} + E_{0r} - E_{0t}] \times \hat{n} = 0$$

and

$$H_i\big|_\perp = H_r\big|_\perp \quad \text{or} \quad [H_{0i} + H_{0r} - H_{0t}] \times \hat{n} = 0$$

$\Rightarrow\ E_{0i} + E_{0r} - E_{0t} = 0$ and $\sqrt{\frac{\varepsilon_0}{\mu_0}}\left(|E_{0i}| - |E_{0r}|\right) = \sqrt{\frac{\varepsilon_s}{\mu_s}} \cdot |E_{0t}|$

$\Rightarrow\ r_s = \dfrac{|E_{0r}|}{|E_{0i}|} = \dfrac{-\mu_s + \sqrt{\mu_s \cdot \varepsilon_s}}{\mu_s + \sqrt{\mu_s \varepsilon_s}}$, μ, ε ambient set to 1, $\varphi_0 = 0$

now use relations: $\varepsilon = \varepsilon_1 - i\varepsilon_2 = (n^2 - k^2) - i(2nk) = \dfrac{c^2}{n^2} - i\dfrac{\sigma\mu c^2}{\omega}$

$\sqrt{\varepsilon} = n - ik$

to separate real- and imaginary parts in r_s

$$\rightsquigarrow\ r_s = \frac{1 - b - ic}{1 + b + ic} = \frac{(1-b-ic)(1+b-ic)}{(1+b+ic)(1+b-ic)} = \frac{1 - b^2 - c^2 - 2ic}{(1+b)^2 + c^2} \Rightarrow$$

Problem #3 continued:

$$\text{lim}\quad r_s = \frac{\sqrt{\varepsilon_s} - \mu_s}{\sqrt{\varepsilon_s} + \mu_s} = \frac{\sqrt{\varepsilon_s^r - i\varepsilon_s^i} - \mu_s}{\sqrt{\varepsilon_s^r - i\varepsilon_s^i} + \mu_s} = \frac{\sqrt{\varepsilon_s^r - 2i/\delta\omega^2} - \mu_s}{\sqrt{\varepsilon_s^r - 2i/\delta\omega^2} + \mu_s}$$

$$\left[\ \varepsilon_s^i = \sigma\mu/\omega = 2/\delta\omega^2\ ,\quad \delta = 2/\mu\sigma\omega :\ = \text{skin depth}\right]$$

use relationship $\sqrt{a+ib} = \sqrt{\frac{a+\sqrt{a^2+b^2}}{2}} + i\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}$

$$\hookrightarrow \sqrt{\varepsilon_s^r - i\varepsilon_s^i} = \sqrt{\frac{\sqrt{\varepsilon_s^{r2} + \varepsilon_s^{i2}} + \varepsilon_s^r}{2}} - i\sqrt{\frac{\sqrt{\varepsilon_s^{r2} + \varepsilon_s^{i2}} - \varepsilon_s^r}{2}}$$

$$\rightsquigarrow b = \frac{1}{\sqrt{\mu_s}} \cdot \sqrt{\frac{\sqrt{\varepsilon_s^{r2} + 4/\delta^2\omega^4} + \varepsilon^{r2}}{2}}\ ;\quad c = \frac{1}{\sqrt{\mu_s}} \cdot \sqrt{\frac{\sqrt{\varepsilon_s^{r2} + 4/\delta^2\omega^4} - \varepsilon_s^r}{2}}$$

$$\text{with } R = r_s \cdot r_s^* = |r_s|^2 \cdot e^{i\theta} = \frac{[(1-b^2-c^2) - 2ic] \cdot [(1-b^2-c^2) + 2ic]}{[(1+b)^2 + c^2]^2}$$

$$\rightsquigarrow \theta = \arctan\left(\frac{-2c}{1-b^2-c^2}\right) = \text{Phase}$$

b.) poor conductor : $\sigma \to 0 \rightsquigarrow \delta \to \infty \Rightarrow c = 0$

$$\hookrightarrow \text{Phase } \theta \to 0 \quad \text{and} \quad r_s = \frac{\sqrt{\varepsilon_s/\mu_s} - 1}{\sqrt{\varepsilon_s/\mu_s} + 1} \approx \underline{\underline{\frac{n-1}{n+1}}}$$

c.) good conductor: $\sigma \to \infty \Rightarrow \delta \to 0$

ε_s^i becomes dominant , we can neglect ε_s^r

$$\hookrightarrow b \approx c \quad \text{and} \quad c \approx \frac{1}{\mu_s\omega} \cdot \sqrt{\frac{\mu_0}{\varepsilon_0}} \cdot \frac{1}{\delta}$$

$$\hookrightarrow R = r_s r_s^* = \frac{[(1-b^2-c^2)-2ic][(1-b^2-c^2)+2ic]}{[(1+b)^2+c^2]^2} \approx \underbrace{\frac{1+4c^4}{4c^4}}_{\approx 1} \cdot \left(1 - \frac{2}{c}\right)$$

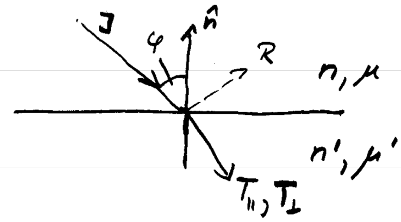
$$\approx 1 - \frac{2}{c} = 1 - 2\left(\frac{\mu_s\omega}{\sqrt{\mu_0/\varepsilon_0}} \cdot \delta\right) = 1 - 2\,\frac{\omega}{c} \cdot \delta$$

Problem #4: An unpolarized light is incident upon a dielectric interface at Brewster's angle. Find the ratio of the transmission coefficient T_1/T_2, and show that this ratio is greater than unity for n not equal to n'. (20 points)

For unpolarized light, we have equal distribution of parallel (∥) and perpendicular (⊥) polarized light in the incoming ray beam.

Dielectric interface: No absorption, $R+T=1$

for $n < n'$ [vacum (n) → dielectric (n')]



$$\varphi_B = \arctan(n'/n)$$

with $\mu = \mu' = 1$, the Fresnel coefficients simplify

$$t_{\parallel} = \frac{2\cdot n\cdot n'\cos\varphi_B}{n'\cos\varphi_B + n\cdot\sqrt{n'^2 - n^2\sin^2\varphi_B}} \quad \text{and} \quad t_s = \frac{2\cdot n\cdot\cos\varphi_B}{n\cdot\cos\varphi_B + \sqrt{n' - n^2\sin^2\varphi_B}}$$

$$t_{\parallel}\big|_{\varphi_B} = \frac{2\cdot n\cdot n'\cdot\frac{n}{\sqrt{n'^2+n^2}}}{\frac{n'^2\cdot n}{\sqrt{n'^2+n^2}} + n\sqrt{n'^2 - \frac{n^2\cdot n'^2}{n'^2+n^2}}} = \ldots \frac{2n^2n'}{2n\,n'^2} = \frac{n}{n'}$$

$$t_{\perp}\big|_{\varphi_B} = \frac{2\cdot n\cdot\frac{n}{\sqrt{n'^2+n^2}}}{n\cdot\left(\frac{n}{\sqrt{n'^2+n^2}}\right) + \sqrt{n' - \frac{n^2\cdot n'^2}{n^2+n'^2}}} = \ldots = \frac{2n^2}{n^2+n'^2}$$

$$\hookrightarrow \quad T_{\perp} = \alpha\cdot\frac{4n^4}{(n^2+n'^2)^2}, \qquad T_{\parallel} = \alpha\cdot\frac{n^2}{n'^2}$$

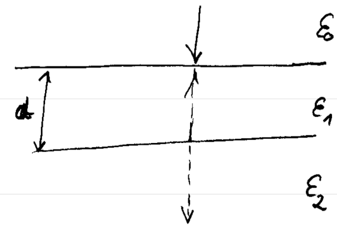
ratio $$\frac{T_{\parallel}}{T_{\perp}} = \frac{\cancel{n^2}\cdot(n^2+n'^2)^2}{n'^2\cdot 4n^{\cancel{4}2}} = \frac{n^4+n'^4+2n^2n'^2}{4n^2\cdot n'^2} > 1 \text{ for } n' > n$$

$$\rightsquigarrow \boxed{T_{\parallel} > T_{\perp}}$$

Problem #5: A thin dielectric film of thickness ***d*** and the dielectric function ε_1 (ε_1 real) lies between media of dielectric functions ε_0 and ε_2. A light wave of frequency ω is incident normally from ε_0. Calculate the reflection coefficient *R*. If $\varepsilon_0 = \varepsilon_2 = 1$, simplify *R* and find the conditions for minimum and maximum reflections as function of film thickness, assuming a fixed wavelength λ. (20 points)

Three layer stack, normal incidence

$\mu_0, \mu_1, \mu_2 = 1$



From matrix formula

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & r_{01} \\ r_{01} & 1 \end{pmatrix} \begin{pmatrix} 1 & r_{12} \\ r_{12}\, e^{-2i\phi_1} & e^{-2i\phi_1} \end{pmatrix} \ldots$$

$$\hookrightarrow \quad r_{123} = \frac{r_{01} + r_{12} \cdot e^{-2i\phi_1}}{1 + r_{01} \cdot r_{12} \cdot e^{-2i\phi_1}} \quad , \quad \phi_1 = \frac{2\pi d_1}{\lambda} \cdot \sqrt{\varepsilon_1 - \varepsilon_0 \sin^2\varphi_0}$$

$$R = r_{123} \cdot r_{123}^* = \frac{r_{01} \cdot r_{01}^* + r_{01} \cdot r_{12} \cdot e^{2i\phi_1} + r_{01}^* \cdot r_{12} \cdot e^{-2i\phi_1} + r_{12} \cdot r_{12}^*}{1 + r_{01}^* r_{12}^*\, e^{2i\phi_1} + r_{01} \cdot r_{12} \cdot e^{-2i\phi_1} + r_{01} \cdot r_{01}^* \cdot r_{12} \cdot r_{12}^*}$$

Normal Incidence; and $\varepsilon_2 = \varepsilon_0$:

$$\rightsquigarrow \quad r_{01} = \frac{\sqrt{\varepsilon_1} - 1}{\sqrt{\varepsilon_1} + 1} \quad , \quad r_{12} = \frac{1 - \sqrt{\varepsilon_1}}{1 + \sqrt{\varepsilon_1}} \quad , \quad \phi_1 = \frac{2\pi d}{\lambda} \cdot \sqrt{\varepsilon_1}$$

dielectric $n = \sqrt{\varepsilon_1}$, all real values

$$\hookrightarrow \quad R = \frac{\frac{(n-1)^2}{(n+1)^2} + \frac{(n-1)(1-n)}{(n+1)^2} \cdot e^{2i\phi_1} + \frac{(n-1)(1-n)}{(n+1)^2} e^{-2i\phi_1} + \frac{(1-n)^2}{(n+1)^2}}{1 + \frac{(n-1)(1-n)}{(n+1)^2} e^{2i\phi_1} + \frac{(n-1)(1-n)}{(n+1)^2} e^{-2i\phi_1} + \frac{(n-1)^2(1-n)^2}{(n+1)^4}}$$

$$= \frac{(n-1)^2 \cdot \left[2 - \left(e^{2i\phi_1} + e^{-2i\phi_1} \right) \right]}{(n+1)^2 - (n-1)^2 \cdot \underbrace{\left[e^{2i\phi_1} + e^{-2i\phi_1} \right]}_{= -2\cos 2\phi_1} + \frac{(n-1)^4}{(n+1)^2}} = \frac{(n-1)^2 \cdot \left[2 + 2\cos 2\phi_1 \right]}{(n+1)^2 + (n-1)^2\, 2\cos 2\phi_1 + \frac{(n-1)^4}{(n+1)^2}}$$

$$\cos 2\phi_1 = \cos\left[2 \cdot \frac{2 \cdot \pi \cdot d}{\lambda} \cdot \sqrt{\varepsilon_1} \right] = \cos\left[\frac{4\pi d}{\lambda} \cdot n \right]$$

$\rightsquigarrow$ <u>Maximum:</u> $\frac{4\pi d}{\lambda} \cdot n = \frac{\pi}{2} \rightsquigarrow \boxed{d_0 = \frac{\lambda}{8 \cdot \sqrt{\varepsilon_1}}};$ <u>Minimum:</u> $\boxed{d_0 = \frac{\lambda}{4\sqrt{\varepsilon_1}}}$