Solutions for Homework # 3

Problem #1 *(30Points)*: Problem 7.23, Jackson textbook (page 348).

The Kamers-Kronig relations are (Jackson equation 7.120):

$$\Re\left(\frac{\epsilon(\omega)}{\epsilon_0}\right) = 1+\frac{2}{\pi}p\int_0^\infty \frac{\omega'\Im\left(\frac{\epsilon(\omega')}{\epsilon_0}\right)}{\omega'^2-\omega^2}d\omega' \quad (1)$$

$$\Im\left(\frac{\epsilon(\omega)}{\epsilon_0}\right) = -\frac{2\omega}{\pi}p\int_0^\infty \frac{\Re\left(\frac{\epsilon(\omega')}{\epsilon_0}-1\right)}{\omega'^2-\omega^2}d\omega' \quad (2)$$

Now, for a plane wave,

$$\vec{E}(\vec{x},t) = \vec{E}_0 e^{i\vec{k}\cdot\vec{x}-i\omega t} \quad (3)$$

$$\vec{B}(\vec{x},t) = \vec{B}_0 e^{i\vec{k}\cdot\vec{x}-i\omega t} \quad (4)$$

Using Ohm's law, $\vec{J}(\vec{x},t)=\sigma\vec{E}(\vec{x},t)$, and Maxwell's $\vec{\nabla}x\vec{H}$ equation,

$$\vec{\nabla}x\vec{H} = \vec{J}+\frac{\partial\vec{D}}{\partial t} \quad (5)$$

$$= \sigma\vec{E}-(-i\omega)\epsilon\vec{E} \quad (6)$$

$$= (-i\omega)\left(\epsilon+i\frac{\sigma}{\omega}\right)\vec{E} \quad (7)$$

Now, from Jackson's equation 7.57, for low frequencies,

$$\sigma(\omega)=\epsilon+\frac{i\sigma}{\omega} \quad (8)$$

Using the hint,

$$\epsilon(\omega)=\epsilon(\omega)+\frac{i\sigma}{\omega}-\frac{i\sigma}{\omega} \quad (9)$$

So, let

$$\epsilon'(\omega)=\epsilon(\omega)-\frac{i\sigma}{\omega} \quad (10)$$

Then $\epsilon'(\omega)$ is analytic for $\Im(\omega)\geq 0$, so I can use it in the Kramers-Kronig realtions.

$$\Re\left(\frac{\epsilon'(\omega)}{\epsilon_0}\right) = 1+\frac{2}{\pi}p\int_0^\infty \frac{\omega'\Im\left(\frac{\epsilon'(\omega')}{\epsilon_0}\right)}{\omega'^2-\omega^2}d\omega' \quad (11)$$

$$\Im\left(\frac{\epsilon'(\omega)}{\epsilon_0}\right) = -\frac{2\omega}{\pi}p\int_0^\infty \frac{\Re\left(\frac{\epsilon'(\omega')}{\epsilon_0}-1\right)}{\omega'^2-\omega^2}d\omega' \quad (12)$$

Now, since $\epsilon'(\omega)=\epsilon(\omega)-\frac{i\sigma}{\omega}$, $\Re\epsilon'(\omega)=\Re\epsilon(\omega)$. Then the second relation can be written (multiplying both sides by ϵ_0),

$$\Im\left(\epsilon(\omega)-i\frac{\sigma}{\omega}\right) = -\frac{2\omega}{\pi}p\int_0^\infty \frac{\Re\left(\epsilon(\omega')-\epsilon_0\right)}{\omega'^2-\omega^2}d\omega' \quad (13)$$

$$\Im\left(\epsilon(\omega)\right) = \frac{\sigma}{\omega}-\frac{2\omega}{\pi}p\int_0^\infty \frac{\Re\left(\epsilon(\omega')-\epsilon_0\right)}{\omega'^2-\omega^2}d\omega' \quad (14)$$

Which is just the expression given in the problem.

Now for the other relation:

$$\Re\left(\frac{\epsilon(\omega)}{\epsilon_0}\right) = 1+\frac{2}{\pi}p\left[\int_0^\infty \frac{\omega'\Im\left(\frac{\epsilon(\omega)}{\epsilon_0}\right)}{\omega'^2-\omega^2}d\omega' - \frac{1}{\epsilon_0}\int_0^\infty \frac{\omega'\frac{\sigma}{\omega'}}{\omega'^2-\omega^2}d\omega'\right] \quad (15)$$

Looking at the last piece of the RHS:

$$\int_0^\infty \frac{d\omega'}{\omega'^2-\omega^2} = \lim_{\omega'\to\infty}\left[\frac{1}{2\omega}\ln\left(\frac{\omega'-\omega}{\omega'+\omega}\right)\right]=0 \quad (16)$$

Where the intgral and limit were evaluated on a calculator. So,

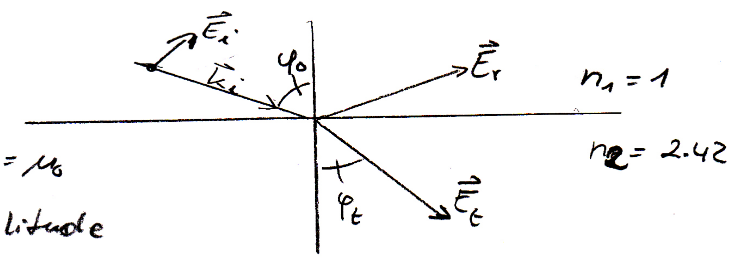
$$\Re\left(\frac{\epsilon(\omega)}{\epsilon_0}\right) = 1+\frac{2}{\pi}p\int_0^\infty \frac{\omega'\Im\left(\frac{\epsilon(\omega')}{\epsilon_0}\right)}{\omega'^2-\omega^2}d\omega' \quad (17)$$

And the first relation remains unchanged from the original.

Problem #2 *(30Points)*: The index of refraction of diamond is $n_2 = 2.42$.

a) Construct a graph that shows the perpendicular and parallel reflected amplitudes as function of angle of incidence for a air -diamond ($n_1 = 1.0$) interface.
b) Construct a similar graph for the perpendicular and parallel reflectance as function of angle of incidence.
c) Calculate the Brewster angle for the air -diamond interface.
d) Calculate the "crossover" angle, at which the reflected and transmitted amplitudes are equal.

a.)

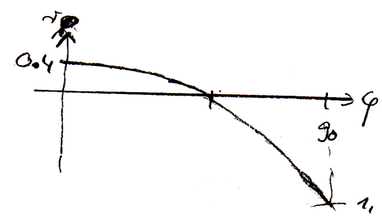


assume $\mu_1 = \mu_2 = \mu_0$

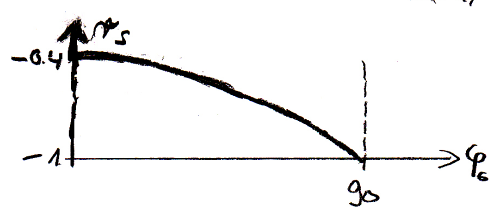
reflected amplitude

$$r_p = \frac{E_{ro}}{E_{io}}\Big|_{\parallel} \quad \text{and} \quad r_s = \frac{E_{rs}}{E_{is}}\Big|_{\perp}$$

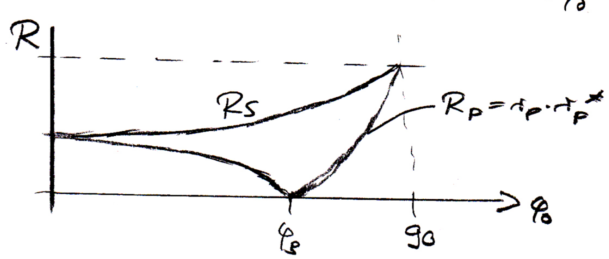
$$\hookrightarrow r_p = \frac{5.86\cos\varphi_0 - \sqrt{5.85 - \sin^2\varphi_0}}{5.86\cos\varphi_0 + \sqrt{5.85 - \sin^2\varphi_0}}$$



$$r_s = \frac{\cos\varphi_0 - \sqrt{5.85 - \sin^2\varphi_0}}{\cos\varphi_0 + \sqrt{5.85 - \sin^2\varphi_0}}$$



b.)



c.) Brewster angle φ_B : $\tan\varphi_B = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{n_2}{n_1} = 2.42$

$\hookrightarrow \varphi_B = \arctan(2.42) = 67.6°$

d.) cross over angle:

condition at which $\frac{E_{ro}}{E_{io}} = \frac{E_{to}}{E_{io}}$

lets define $\alpha = \frac{\sqrt{1 - (n_1/n_2)^2 \cdot \sin^2\varphi_0}}{\cos\varphi_0}$ and $\beta = n_2/n_1$

$\rightsquigarrow \frac{\alpha - \beta}{\alpha + \beta} = \frac{2}{\alpha + \beta} \rightsquigarrow \alpha - \beta = 2 = \ldots = 4.42 \rightsquigarrow \cos\varphi_c = 0.207$

or $\varphi_c = 78.1°$

Problem #3: The phenomenon that the permittivity ε changes as function of frequency is called dispersion. By extension, whenever the speed of a wave depends on its frequency, the supporting medium is called dispersive! Shallow water ($d < \lambda$) is nondispersive; the waves travel at a speed that is proportional to the square root of the depth. Show that the wave velocity v is twice the group velocity $v_g = d\omega/dk$. *(20 Points)*

If the depth "d" of water is much smaller than λ: $d \ll \lambda$, the phase velocity is proportional to $\sqrt{d}$

$$\rightsquigarrow \quad v_p = g \cdot \sqrt{d}$$

v_p does not depend on wavelength λ

$\rightsquigarrow$ non-dispersive propagation with

$$v_p = v_g = \text{const.}$$

For $d \gg \lambda$:

$$v_p = g \cdot \frac{\lambda}{2\pi} \overset{!}{=} \frac{\omega}{k}$$

$\rightsquigarrow$ dispersion relation $\underline{\underline{\omega^2 = g^2 \cdot k}}$

$$v_g = \frac{d\omega}{dk} \Rightarrow \boxed{v_p = 2 \cdot v_g}$$

Problem #4: Assuming negligible damping ($\gamma_j = 0$), calculate the group velocity ($v_g = d\omega/dk$) of waves described by $\vec{E}(z,t) = \vec{E}_0 \cdot e^{-\kappa \cdot z} \cdot e^{i(k \cdot z - w \cdot t)}$, ... (20 Points)

group velocity $v_g = \frac{d\omega}{dk}$

wave: $\vec{E}(z,t) = \vec{E}_0 \cdot e^{-kz} \cdot e^{i(k \cdot z - \omega t)}$

and dielectric function $\varepsilon(\omega) = 1 + \frac{N \cdot q^2}{2m\varepsilon} \sum \frac{f_i}{\omega_i^2 - \omega^2 - i\Gamma_i\omega}$

For negligible damping: $\Gamma_i = 0 \rightsquigarrow \varepsilon = n^2$ or $n = \sqrt{\varepsilon}$

$$k = \frac{\omega}{c}\sqrt{1 + \frac{Nq^2}{2m\varepsilon_0}\sum_i \frac{f_i}{\omega_i^2 - \omega^2}}$$

$$\hookrightarrow v_g = \frac{d\omega}{dk} = \frac{1}{\frac{n}{c} + \frac{\omega}{c} \cdot \left(\frac{dn}{d\omega}\right)} = \frac{c}{n(\omega) + \omega \cdot \frac{dn}{d\omega}}$$

for $n = 1 \rightsquigarrow v_g = c$; otherwise $v_g < c$

phase velocity $v_p = \frac{\omega(k)}{k} = \frac{c}{n(\omega)}$

for normal dispersion $(n > 1)$: $\frac{dn}{d\omega} > 0$

and with it: $c > v_p > v_g$

for $n < 1$ and $\frac{dn}{d\omega} > 0$

$$\underline{v_p > c > v_g}$$