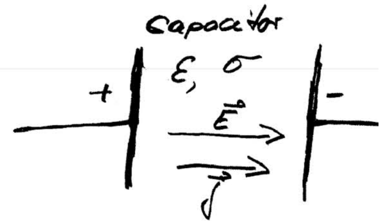
Physics 8110 - Electromagnetic Theory II

Solutions for Homework # 2

Problem #1: A parallel plate capacitor was charged and then disconnected from the battery. It is discharged due to finite conductivity σ of the media between the plates. The dielectric permittivity is ε. Find the conduction current density and the total current density inside the capacitor as a function of time. Find the flux of the electromagnetic energy from the space between the plates. *(20 Points)*

given are σ and ε



a.) find conduction $\vec{j}_c$ and total current $\vec{j}$!

electric field $\vec{E} = \frac{\sigma}{\varepsilon} = \frac{Q}{\varepsilon \cdot A}$

conduction $j_c = \sigma \cdot \vec{E} = \frac{\sigma \cdot Q(t)}{\varepsilon \cdot A}$

change of charge $\underline{\frac{dQ}{dt}} = -\int_S \vec{j} \cdot d^2\vec{r} = -\vec{j} \cdot A = \underline{\frac{-\sigma \cdot Q}{\varepsilon}}$

$\hookrightarrow Q(t) = Q_0 \cdot e^{-t \cdot \sigma/\varepsilon} = Q_0 \cdot e^{-t/\tau}; \quad \tau = \varepsilon/\sigma$

$\hookrightarrow \underline{j_c = \frac{\sigma}{\varepsilon \cdot A} \cdot Q_0 \cdot e^{-t/\tau}}$

Diffusion current $\underline{j_D} = \varepsilon_0 \frac{\partial E}{\partial t} = \frac{1}{A} \frac{\partial Q}{\partial t} = \underline{\underline{-\frac{Q_0}{A \cdot \tau} \cdot e^{-t/\tau}}}$

$\hookrightarrow \vec{j}_c + \vec{j}_D = 0$ = total current

$\hookrightarrow \vec{B} = 0$

b.) Flux $S = \frac{1}{\mu}(E \times \vec{B}) = \underline{\underline{0}}$

Problem #2: Derivate from the Maxwell equations and the scalar and vector potential definitions the wave propagation equations …. *(20 Points)*

Start with Maxwell's equations:

$$\nabla \cdot \vec{D} = \rho, \quad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{H} = \vec{j} + \frac{\partial D}{\partial t}, \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Vector potential

$$\vec{B} = \nabla \times \vec{A}$$

$$\nabla \times \vec{E} = -\frac{\partial B}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \vec{A}) \rightsquigarrow \nabla \times \left(E + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

$$\hookrightarrow \vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t}$$

integrated in $\nabla \times \vec{H} = \vec{j} + \frac{\partial}{\partial t}(\varepsilon \cdot \vec{E})$

$$= \vec{j} + \varepsilon \cdot \left(-\nabla \frac{\partial \phi}{\partial t} - \frac{\partial^2 \vec{A}}{\partial t^2}\right) = \vec{j} - \varepsilon \nabla \dot{\phi} - \varepsilon \ddot{\vec{A}}$$

replace $\nabla \times \vec{H} = \frac{1}{\mu}(\nabla \times \vec{B}) = \frac{1}{\mu}(\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

$$\hookrightarrow \frac{1}{\mu}\left(\nabla(\nabla \cdot \vec{A}) - \Delta \vec{A}\right) = \vec{j} - \varepsilon \nabla \dot{\phi} - \varepsilon \ddot{\vec{A}}$$

$$\rightsquigarrow \boxed{\Delta \vec{A} - \nabla[\nabla \cdot \vec{A} + \mu \varepsilon \dot{\phi}] - \mu \cdot \varepsilon \ddot{\vec{A}} = -\mu \cdot \vec{j}}$$

Problem #3: Dispersion relations: Problem 7.20 (a) thru (c), Jackson textbook (page 347).
(30 Points)

a.) For a homogeneous, isotropic media with $\mu = 1$, a plane wave has to satisfy the Telegraph equation:

$$\left[\Delta - \mu\sigma - \mu\varepsilon\right]\left\{\begin{matrix}\vec{E}\\ \vec{B}\end{matrix}\right\} = 0$$

↳ One dim. solution: $u(\omega, n, t) = \left[A(\omega)\, e^{\frac{i\omega \cdot n(\omega)\cdot k}{c}} + B(\omega)\, e^{-\frac{i\omega\cdot n(\omega) k}{c}}\right] e^{-i\omega t}$

use Fourier-transform $\omega \rightarrow x$ to get

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega t}\cdot\left[A(\omega)\, e^{i\omega\cdot n\cdot k/c} + B(\omega)\, e^{-i\omega\cdot n\cdot k/c}\right] d\omega$$

$n(\omega) = \sqrt{\frac{c^2}{v^2} - i\mu\sigma c^2/\omega}$ from Telegraph equ.

b.) $n^*(\omega) = \sqrt{c^2/v^2 + i\mu\sigma c^2/\omega} = \sqrt{c^2/v^2 - i\mu\sigma c^2/(-\omega)} = \underline{\underline{n(-\omega)}}$

c.) put $x = 0$ in $u(x,t)$ in a.)

$$u(0,t) = \frac{1}{\sqrt{2\pi}}\cdot\int_{-\infty}^{\infty}\left[A(\omega) + B(\omega)\right] e^{-i\omega t}\, d\omega$$

$$\frac{\partial u(0,t)}{\partial x} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[A(\omega) - B(\omega)\right]\frac{i\cdot\omega\cdot n}{c}\cdot e^{-i\omega t}\, d\omega$$

↳ $A(\omega') + B(\omega') = \frac{1}{\sqrt{2\pi}}\int u(0,t)\, e^{i\omega' t}\, dt$

$A(\omega') - B(\omega') = \frac{1}{\sqrt{2\pi}}\int \frac{-i c}{\omega'\cdot n}\,\frac{\partial u(0,t)}{\partial x}\, e^{i\omega' t}\, dt$

↳ $A(\omega') = \frac{1}{2\sqrt{2\pi}}\int\left[u(0,t) - \frac{i\cdot c}{\omega'\cdot n}\frac{\partial u(0,t)}{\partial x}\right] e^{i\omega' t}\, dt$

and $B(\omega') = \frac{1}{2\sqrt{2\pi}}\int\left[u(0,t) + \frac{i c}{\omega' n}\frac{\partial u(0,t)}{\partial x}\right] e^{i\omega' t}\, dt$

$$\Rightarrow \left\{\begin{matrix}A(\omega)\\ B(\omega)\end{matrix}\right\} = \frac{1}{2\cdot\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[u(0,t) \mp \frac{ic}{\omega\cdot n}\frac{\partial u(0,t)}{\partial x}\right] e^{i\omega t}\, dt \;//$$

Problem #4: Kramers-Kronig: Problem 7.22 (a) and (b), Jackson textbook (page 348). *(30 Points)*

Kramers-Kronig

$$\varepsilon_1(\omega) = 1 + \frac{2}{\pi} P\int_0^\infty \frac{\omega' \cdot \varepsilon_2(\omega')}{\omega'^2 - \omega^2}\, d\omega'$$

and

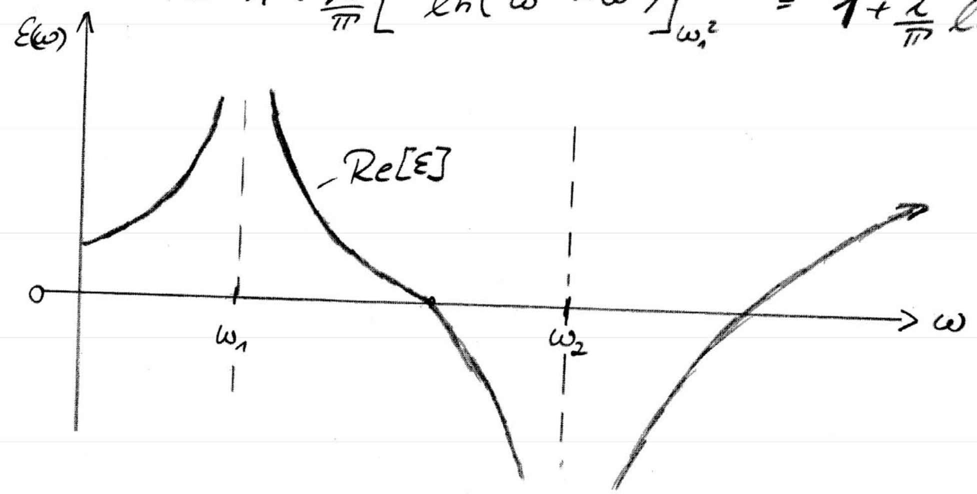
$$\varepsilon_2(\omega) = -\frac{2\omega}{\pi} P\int_0^\infty \frac{\varepsilon_1(\omega') - 1}{\omega'^2 - \omega^2}\, d\omega'$$

a) $\varepsilon_2(\omega) = \lambda \cdot [\Theta(\omega - \omega_1) - \Theta(\omega - \omega_2)]$ with $\omega_1, \omega_2 > 0;\ \omega_2 > \omega_1$

$$\rightarrow \varepsilon_1(\omega) = 1 + \frac{2\cdot\lambda}{\pi}\int_{\omega_1}^{\omega_2} \frac{\omega'}{\omega'^2 - \omega^2}\, d\omega'$$

$$= 1 + \frac{2\cdot\lambda}{\pi}\cdot\left(\frac{1}{2}\int_{\omega_1^2}^{\omega_2^2} \frac{d(\omega'^2)}{\omega'^2 - \omega^2}\right)$$

$$= 1 + \frac{\lambda}{\pi}\left[\ln(\omega'^2 - \omega^2)\right]_{\omega_1^2}^{\omega_2^2} = 1 + \frac{\lambda}{\pi}\ln\left(\frac{\omega_2^2 - \omega^2}{\omega_1^2 - \omega^2}\right)$$



b.) $\varepsilon_2(\omega) = \frac{\lambda\cdot\gamma\cdot\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \rightarrow \varepsilon_1(\omega) = 1 + \frac{2}{\pi} P\int_0^\infty \frac{\omega'\cdot\varepsilon_2(\omega')}{\omega'^2 - \omega^2}\, d\omega'$

$$\varepsilon_1 = 1 + \frac{2}{\pi}\int_0^\infty \frac{\omega'\cdot\lambda\cdot\gamma\cdot\omega}{(\omega'^2 - \omega^2)\cdot[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2]}\, d\omega' \overset{\text{Hilbert transformation}}{=} 1 + \frac{\lambda\cdot(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$