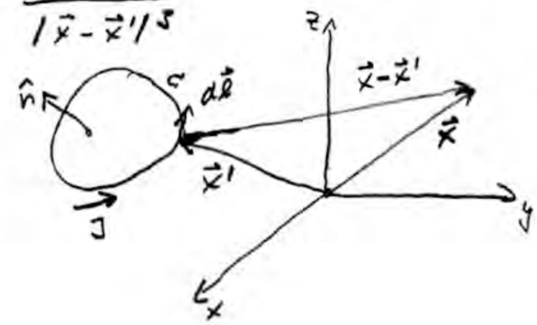
Solutions for Homework # 1

Problem #1: Problem 5.1, Jackson textbook *(10 Points)*

Starting with $d\vec{B} = \frac{\mu_0 J}{4\pi} d\vec{l}' \times \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$



$$\vec{B} = \oint d\vec{B} = \oint_C \frac{\mu_0 J}{4\pi} d\vec{l}' \times \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

$$= \frac{\mu_0 J}{4\pi} \oint_C d\vec{l}' \times \vec{\nabla} \frac{1}{|\vec{x}-\vec{x}'|}$$

$$= \frac{\mu_0 J}{4\pi} \int_S (d\vec{a}' \times \vec{\nabla}') \times \vec{\nabla} \frac{1}{|\vec{x}-\vec{x}'|}$$

$$= \frac{\mu_0 J}{4\pi} \int_S \left[\vec{\nabla}' d\vec{a}' \cdot \nabla \frac{1}{|\vec{x}-\vec{x}'|} - d\vec{a}' \cdot \vec{\nabla}' \cdot \nabla \frac{1}{|\vec{x}-\vec{x}'|} \right]$$

$$= -\frac{\mu_0 J}{4\pi} \vec{\nabla}' \int d\vec{a}' \cdot \nabla \frac{1}{|\vec{x}-\vec{x}'|} = -\frac{\mu_0 J}{4\pi} \vec{\nabla}' \cdot \int d\vec{a} \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

$$= \frac{\mu_0 J}{4\pi} \cdot \vec{\nabla}' \cdot \int d\vec{a}\ \hat{n}' \cdot \frac{\hat{r}}{r^2} = \frac{\mu_0 J}{4\pi} \vec{\nabla}' \cdot \int \underbrace{da \frac{\cos\theta}{r^2}}_{d\Omega}$$

$$= \frac{\mu_0 J}{4\pi} \nabla' \Omega = \underline{\frac{\mu_0 J}{4\pi} \nabla \Omega}$$

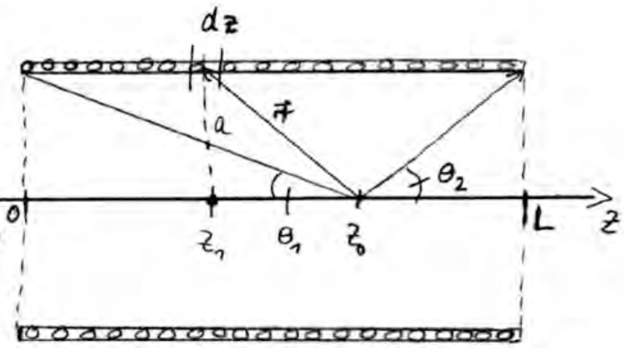
Problem #2: Jackson textbook - Problem 5.3 *(15 Points)*

N - turns per unit length

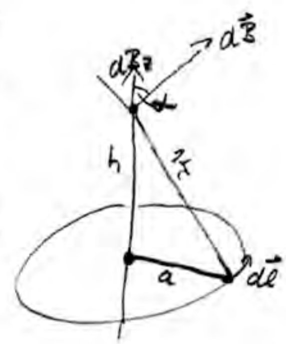
Current J

Show that $B_z = \frac{\mu_0 N J}{2}(\cos\theta_1 + \cos\theta_2)$

in the limit $NL \to \infty$



a) B_z of a circular loop with current J:



$$d\vec{B} = \frac{\mu_0}{4\pi} \cdot J \, \frac{d\vec{\ell} \times \vec{r}}{r^3}, \quad r = \sqrt{h^2 + a^2}$$

$$dB_z = d\vec{B} \cdot \cos\alpha = d\vec{B} \cdot \frac{a}{r} = \frac{\mu_0}{4\pi} J \frac{d\ell}{r^2} \cdot \frac{a}{r}$$

$$B_z = \int dB_z = \frac{\mu_0}{4\pi} J \cdot \frac{2\pi a}{r^3} \cdot a = \frac{\mu_0 \cdot J a^2}{2 r^3}$$

b) element of solenoid $(z_1, z_1 + dz)$ - circular loop:

$$dB_z = \frac{\mu_0 a^2}{2 r^3} dz \, N \cdot J \;; \quad r = \sqrt{(z_0 - z_1)^2 + a^2}$$

$$= \underbrace{\frac{\mu_0 a^2 N J}{2}}_{= A} \cdot \frac{dz}{\sqrt{a^2 + (z_0 - z_1)^2}^{\,3}}$$

$$B_z = \int dB_z = A \cdot \int_0^L \frac{dz}{\sqrt{a^2 + (z_0 - z_1)^2}^{\,3}} = A \cdot \int_{-z_0}^{L - z_0} \frac{dy}{\sqrt{1 + (y/a)^2}^{\,3} \cdot a^3}$$

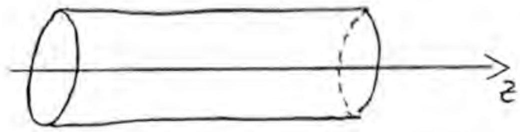
substitute $y = z - z_0$

$$= A \cdot \frac{1}{a^2} \int_{-z_0/a}^{(L - z_0)/a} \frac{dx}{\sqrt{1 + x^2}^{\,3}} = \frac{\mu_0 N J_0}{2} \frac{x}{\sqrt{1 + x^2}} \Bigg|_{-z_0}^{L - z_0}$$

$$= \frac{\mu_0 N \cdot J}{2} \left[\frac{(L - z_0)/a}{\sqrt{1 + \left(\frac{L - z_0}{a}\right)^2}} + \frac{z_0/a}{\sqrt{1 + (z_0/a)^2}} \right] = \frac{\mu_0 N J}{2} (\cos\theta_1 + \cos\theta_2)$$

Problem #3: Find the vector potential of an infinite solenoid with *N* turns per unit length, radius *R*, and current *I*. *(15 Points)*

Infinite solenoid:



$$\left.\begin{array}{l} B_z = \mu_0 J \cdot N \\ B_x = B_y = 0 \end{array}\right\} \text{inside}$$

$B_z = B_x = B_y = 0$ - outside

$$\vec{B} = \vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

1.) <u>inside:</u>

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = 0$$

$$B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = 0$$

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \mu_0 \cdot J \cdot N$$

We can satisfy this equations by:

$A_y = \mu_0 N \cdot J \cdot x$, $A_x = A_z = 0$

or: $A_y = 0$, $A_x = -\mu_0 N \cdot J \cdot y$, $A_z = 0$

$\rightsquigarrow$ Two possible solutions $\vec{A}_1 = (0, \mu_0 N \cdot J \cdot x, 0)$

$\vec{A}_2 = (-\mu_0 N \cdot J \cdot y, 0, 0)$

2. outside: $\vec{B} = 0 \rightsquigarrow$ select $\vec{A} = \vec{0} = (0, 0, 0)$

Problem #3: continued

in cylindrical coordinates (r, θ, z)

We have

$$\oint \vec{A}\cdot d\vec{l} = \int_A (\nabla \times \vec{A})\, d^2r = \int \vec{B}\cdot d^2r = \Phi$$

The magnetic field inside the solenoid is uniform

$$\oint_S \vec{A}\, d\vec{s} = \int_A \vec{B}\, d^2r$$

$$A \cdot (2\pi r) = \mu_0\, N \cdot J \cdot (\pi r^2)$$

$$\rightsquigarrow \boxed{\vec{A} = \frac{\mu_0\, N \cdot J \cdot r}{2}\, \hat{\theta}} \qquad \underline{\text{for } r < R}$$

for $r \geq R$:

$$\oint \vec{A}\cdot d\vec{s} = \int_A \vec{B}\, d^2r$$

$$A \cdot 2\pi r = \mu_0\, N \cdot J\, (\pi R^2)$$

$$\rightsquigarrow \boxed{\vec{A} = \frac{\mu_0\, N \cdot J\, R^2}{2}\left(\frac{1}{r}\right)\hat{\theta}} \qquad (r > R)$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \hat{r}\left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right) + \hat{\theta}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{z}\left(\frac{1}{r}\frac{\partial}{\partial r}(r A_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right)$$

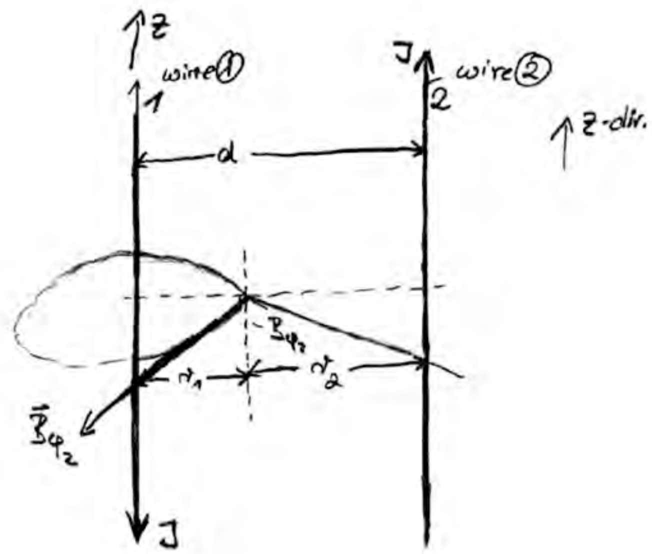
$$= \Phi$$

Problem #4: Find the magnetic vector potential at a point between two long, straight wires carrying the same current *I*, in opposite directions. *(15 Points)*

(A) $\oint \vec{B}\, d\vec{l} = \mu_0 J$

$\rightsquigarrow B_\varphi \cdot 2\pi r = \mu_0 J$

$$B_\varphi = \frac{\mu_0 J}{2\pi r}$$



$$B_{\varphi_1} = -\frac{\mu_0 J}{2\pi r_1}$$

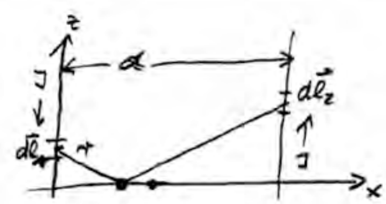
$$B_{\varphi_2} = -\frac{\mu_0 J}{2\pi r_2}$$

Between wires: $B_\varphi = B_{\varphi_1} + B_{\varphi_2} = -\frac{\mu_0 J}{2\pi}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$

$$\vec{B} = \vec{\nabla}\times\vec{A} \rightsquigarrow B_\varphi = \frac{\partial A_z}{\partial r}\cdot\hat{\varphi} = \frac{\mu_0 J}{2\pi}\left(-\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{\mu_0 J}{2\pi}\left(\frac{1}{r} - \frac{1}{(d-r)}\right)$$

$$A_z = \frac{\mu_0 J}{2\pi}\int\left(\frac{1}{r} + \frac{1}{d-r}\right)dr = \frac{\mu_0 J}{2\pi}\left[-\ln(d-r) + \ln(r)\right] + C$$

$$= \frac{\mu_0 J}{2\pi}\ln\left(\frac{r}{d-r}\right) + C$$



(B) second method: $\vec{A} = \mu_0/4\pi \int \frac{J\, d\vec{l}}{|\vec{x}-\vec{x}'|}$

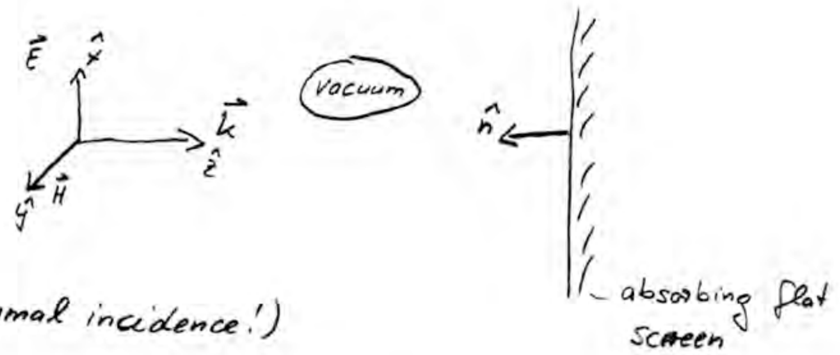
$$\rightsquigarrow \vec{A} = \frac{\mu_0 J}{4\pi}\int_{-\infty}^{\infty}\frac{dz_1\,\hat{k}}{|\vec{r}_1|} + \int_{-\infty}^{\infty}\frac{dz_2\,\hat{k}}{|\vec{r}_2|}$$

$$= \frac{\mu_0 J}{4\pi}\left[-\int\frac{dz_1\,\hat{k}}{\sqrt{r^2+z_1^2}} + \int\frac{dz_2\,\hat{k}}{\sqrt{r^2+z_2^2}}\right] = \frac{\mu_0 J}{4\pi}\left[\ln\left(z+\sqrt{z^2+r^2}\right) + \ln\left(z+\sqrt{z^2+(d-r)^2}\right)\right]$$

$$= \frac{\mu_0 J}{4\pi}\left[-\ln\left[\frac{L+\sqrt{L^2+r^2}}{-L+\sqrt{L^2+r^2}}\right] + \ln\left[\frac{L+\sqrt{L^2+(d-r)^2}}{-L+\sqrt{L^2+(d-r)^2}}\right]\right] \approx \frac{\mu_0 J}{2\pi}\ln\left(\frac{r}{d-r}\right)$$

$(L\to\infty)$

Problem #5: Jackson textbook = Problem 6.11 *(15 Points)*



$\vec{k} \parallel \hat{n}$ (normal incidence!)

a.) Density of EM - momentum:

$$\vec{g} = \mu_0 \varepsilon_0 \vec{S} = \mu_0 \varepsilon_0 \vec{E} \times \vec{H}$$

Momentum deposited on area A of the screen during the time Δt: $\vec{P} = \vec{g} \cdot V = \vec{g} \cdot A \cdot c \cdot \Delta t$ $\left(c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \text{ speed of light} \right)$

$$\leadsto \vec{F} = \frac{\vec{P}}{\Delta t} = \vec{g} \frac{A \cdot c \cdot \Delta t}{\Delta t} = \vec{g} \cdot A \cdot c$$

pressure: $P = \frac{\vec{F}}{A} = \vec{g} \cdot c = \frac{1}{c} \vec{E} \times \vec{H} = \frac{1}{c} |\vec{E}| \cdot |\vec{H}| \cdot \hat{n}$

In transverse plane wave: $H = \sqrt{\frac{\varepsilon_0}{\mu_0}} \vec{E}$ $\leadsto$

$$\frac{F}{A} = \sqrt{\mu_0 \varepsilon_0} \cdot \frac{1}{2} \left[E \cdot \sqrt{\frac{\varepsilon_0}{\mu_0}} \cdot E + \sqrt{\frac{\mu_0}{\varepsilon_0}} \cdot H^2 \right] = \frac{1}{2} \left(\varepsilon_0 E^2 + \mu_0 H^2 \right)$$

$$= \frac{1}{2} \left(\vec{E} \cdot \vec{D} + \vec{H} \cdot \vec{B} \right) = \text{energy density of the field!} \Rightarrow$$

Problem #5: continued

Solution of (a) using stress tensor:

$$F_z = -\frac{dp_{field}}{dz} = -\int_S \sum_\beta T_{z\beta}\, da \quad \Big| \; S := \text{area } A \text{ of the screen}$$

$$T_{z\beta} = \varepsilon_0 \left(E_z E_\beta + c^2 B_z \cdot B_\beta - \frac{1}{2}(E^2 + c^2 B^2)\,\delta_{z\beta} \right)$$

The transverse field: $B_z = 0$, $E_z = 0 \rightsquigarrow T_{z\beta} = -\frac{1}{2}\varepsilon_0 (E^2 + c^2 B^2)\,\delta_{z\beta}$

$$F_z = -A \sum_\beta \left[-\frac{1}{2}\varepsilon_0 (E^2 + c^2 B^2)\,\delta_{z\beta} \right] = \frac{1}{2} A \varepsilon_0 (E^2 + c^2 B^2)$$

Pressure $P = \frac{F_z}{A} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) = \frac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}\right)$ = energy density

(b) Flux of EM energy: $S = 1.4\ \mathrm{kW/m^2}$

$\rho = 1\,\mathrm{g/m^2}$, $a_{max} = ?$

$$\vec{S} = \vec{E}\times\vec{H}, \quad \rightsquigarrow S = E\cdot H = E^2 \cdot \sqrt{\frac{\varepsilon_0}{\mu_0}} = \varepsilon_0 E^2 \cdot \frac{1}{\sqrt{\mu_0 \varepsilon_0}} =$$

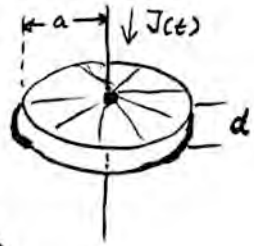
$$= c \cdot \underbrace{\frac{1}{2}(E\cdot D + B\cdot H)}_{= u \text{ - energy density}}$$

Pressure $P = \frac{1}{2}(E\cdot D + B\cdot H) = u = \frac{S}{c}$

$$a_{max} = \frac{P\cdot A}{\rho\cdot A} = \frac{P}{\rho} = \frac{S/c}{\rho} = \frac{1.4\cdot 10^3\ \mathrm{W/m^2}}{3\cdot 10^8\,\frac{m}{s}\cdot 10^{-3}\ \mathrm{kg/m^2}}$$

$$= \frac{1.4}{3}\cdot 10^{-3}\,\frac{m}{s^2} \approx \underline{\underline{4.66\cdot 10^{-3}\,\frac{m}{s^2}}}$$

Problem #6: Jackson textbook = Problem 6.14 (a) *(20 Points)*



Circular parallel plate capacitor with radius a and plate separation $d \ll a$

a.) $J(t) = J_0 \cos\omega t$; $\vec{E}$ and $\vec{B}$ = ?

1.) we neglect the effect of $\vec{B}$ on $\vec{E}$:

$$J(t) = \frac{dq}{dt} \rightsquigarrow q(t) = \int J(t)\,dt = \frac{1}{\omega} J_0 \sin(\omega t)$$

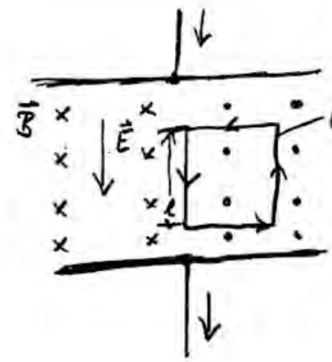
(charge on capacitor)

$$\sigma(t) = \frac{q(t)}{A} = \frac{\frac{1}{\omega} J_0 \sin\omega t}{\pi a^2}$$

$$E(t) = \frac{\sigma}{\varepsilon_0} = \frac{J_0}{\varepsilon_0 \omega \pi a^2} \cdot \sin(\omega t)$$

Displacement current density

$$J_d = \varepsilon_0 \frac{\partial E(t)}{\partial t} = \frac{J_0}{\pi a^2} \cos\omega t$$



$$\oint_C B \cdot d\ell = B \cdot 2\pi r = \mu_0 J_{enclosed} = \mu_0 \pi r^2 \cdot J_d$$

$$= \mu_0 \pi r^2 \cdot \frac{J_0}{\pi \cdot a^2} \cdot \cos\omega t$$

$$\rightsquigarrow B(t) = \mu_0 \frac{r^2}{a^2} J_0 \cos\omega t \cdot \frac{1}{2\pi r} = \frac{\mu_0 r}{2\pi a^2} \cdot J_0 \cos\omega t$$

2.) Now - changing magnetic field induces electric field (Faraday's law)

$\rightsquigarrow E'$ - induced by $\partial\vec{B}/\partial t$:

$$\underbrace{\oint_C \vec{E}' \cdot d\vec{s}}_{= [E'(0) - E'(r)] \cdot \ell} = -\frac{d\phi}{dt} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{a} = -\frac{d}{dt} \ell \cdot \int_0^r B(r)\,dr = \frac{\mu_0 r^2 J_0 \sin(\omega t) \cdot \omega \cdot \ell}{4\pi a^2}$$

$\Longrightarrow$

Problem #6: continued

$$\rightsquigarrow E'(r) = E'(0) - \frac{\mu_0 r^2 \cdot J_0 \cdot \omega \cdot \sin(\omega t)}{4\pi a^2}$$

from $\varepsilon_0 \frac{d}{dt} \int_0^a E'(r) \cdot 2\pi \cdot r \, dr = 0 \rightsquigarrow E'(0) = \mu_0 \cdot J_0 \cdot \omega \cdot \sin(\omega t)/8\pi$

$$\rightsquigarrow \boxed{E'(r) = \frac{\mu_0 \cdot J_0 \, \omega \cdot \sin(\omega t)}{8\pi}\left(1 - \frac{2 \cdot r^2}{a^2}\right)}$$

3.) Now: electric field $\vec{E}'$ induces magnetic field $\vec{B}'$:

$$B' \cdot 2\pi r = \mu_0 \int_0^r \underbrace{\varepsilon_0 \frac{\partial E'}{\partial t}}_{J_d} \cdot 2\pi r \, dr = \frac{\varepsilon_0 \mu_0^2 \omega^2 \cos\omega t}{8\pi}\left(\pi r^2 - \frac{\pi r^4}{a^2}\right)$$

$$\rightsquigarrow \boxed{B'(r) = \frac{\varepsilon_0 \mu_0^2 \omega^2 J_0 \cos(\omega t)}{16\pi} \cdot \left(r - \frac{r^3}{a^2}\right)}$$

4.) Now B' induces E'' - but $E'' \sim \omega^3 \rightsquigarrow$ we can drop E'' and B''....

since we need to find E and B up to ω^2

$$\Rightarrow E(t,r) = \frac{J_0}{\omega \varepsilon_0 \pi a^2} \sin(\omega t) + \frac{\mu_0 J_0 \cdot \omega \cdot \sin(\omega t)}{8\pi}\left(1 - \frac{2r^2}{a^2}\right)$$

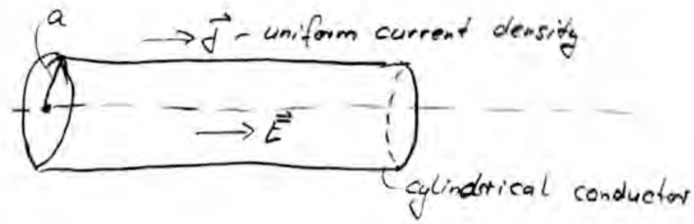
$$= \frac{J_0 \cdot \sin(\omega t)}{\varepsilon_0 \cdot \omega \cdot \pi \cdot a^2}\left[1 + \frac{\varepsilon_0 \mu_0 a^2 \omega^2}{8}\left(1 - \frac{2r^2}{a^2}\right)\right]$$

$$\boxed{E(t,r) = \frac{J_0 \sin(\omega t)}{\varepsilon_0 \omega \cdot \pi a^2}\left[1 + \frac{a^2 \omega^2}{8 \cdot c^2}\left(1 - \frac{2r^2}{a^2}\right)\right]}$$

$$B(t,r) = B + B' = \frac{\mu_0 r J_0}{2\pi a^2} \cdot \cos(\omega t) + \frac{\varepsilon_0 \mu_0^2 \omega^2 J_0 \cdot \cos(\omega t)}{16\pi}\left(r - r^3/a^2\right)$$

$$\boxed{B(t,r) = \frac{\mu_0 \cdot r J_0 \cos(\omega t)}{2\pi a^2} \cdot \left[1 + \frac{\omega^2 a^2}{8c^2}\left(1 - \frac{r^2}{a^2}\right)\right]}$$

Problem #7: Calculate the Poynting vector and its divergence for an infinitely long cylindrical conductor with radius *a* carrying uniform current density *J* due to a uniform electric field *E* parallel to the axis of the conductor inside the conductor. *(10 Points)*



$$\oint_C \vec{B}\, d\vec{l} = 2\pi r \cdot B = \mu_0 I_{encl.} = \mu_0 \pi r^2 \cdot \vec{J}$$

$$\rightsquigarrow B(r) = \frac{\mu_0 \pi r^2 \vec{J}}{2\pi r} = \frac{\mu_0 \cdot r}{2} \cdot \vec{J}$$

$$\vec{S} = \vec{E} \times \vec{H} = \frac{1}{\mu_0} \vec{E} \times \vec{B}, \quad \text{in cylindrical coord. } (r, \varphi, z) \quad S_\varphi = S_z = 0$$

$$S_r = \frac{E \cdot B}{\mu_0} = \frac{E \mu_0 \vec{J} \cdot r}{2\mu_0} = \frac{E \cdot \vec{J} \cdot r}{2}$$

$$\nabla \vec{S} = \frac{1}{r} \frac{\partial}{\partial r}\left(r S_r\right) + \frac{1}{r} \frac{\partial S_\varphi}{\partial \varphi} + \frac{\partial S_z}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r}\left(r S_r\right)$$

$$= \frac{1}{r} \frac{\partial}{\partial r}\left(r \cdot \frac{E j \cdot r}{2}\right) = \frac{1}{r} \frac{E \cdot j}{2} \cdot 2r = E \cdot j = \text{constant} \rightsquigarrow \underline{\text{Joule heat}}$$