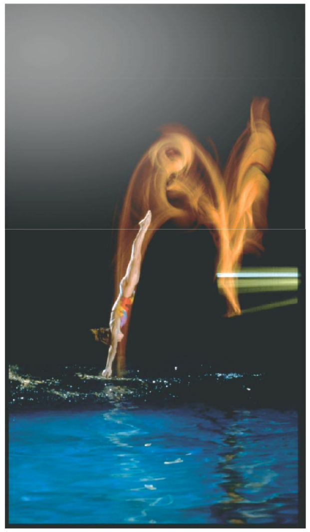
Chapter 4. Kinematics in Two Dimensions

A car turning a corner, a basketball sailing toward the hoop, a planet orbiting the sun, and the diver in the photograph are examples of two-dimensional motion or, equivalently, motion in a plane.

Chapter Goal: To learn to solve problems about motion in a plane.



Chapter 4. Kinematics in Two Dimensions

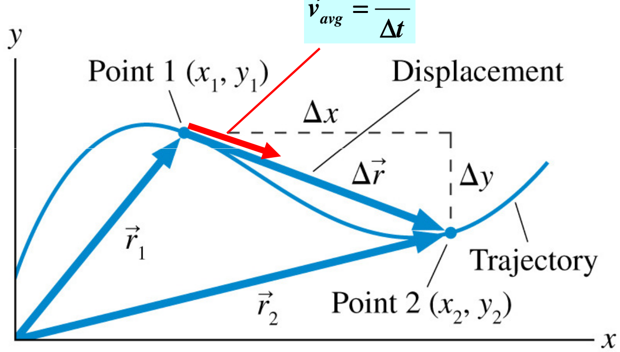
Topics:

- Acceleration
- Kinematics in Two Dimensions
- Projectile Motion
- Relative Motion
- Uniform Circular Motion
- Velocity and Acceleration in Uniform Circular Motion
- Nonuniform Circular Motion and Angular Acceleration

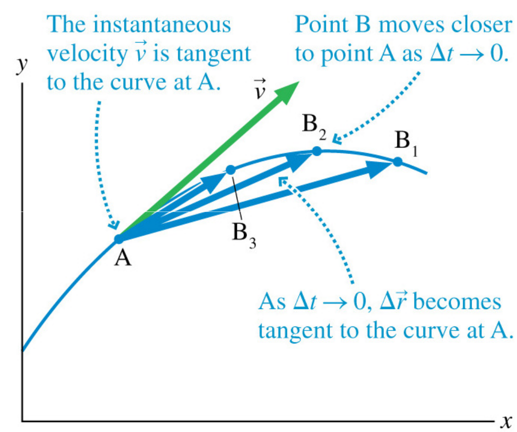
Kinematics in Two Dimensions

Average velocity: $\vec{v}_{avg} = \dfrac{\Delta \vec{r}}{\Delta t}$

$\vec{v}_{avg} = \dfrac{\Delta \vec{r}}{\Delta t}$

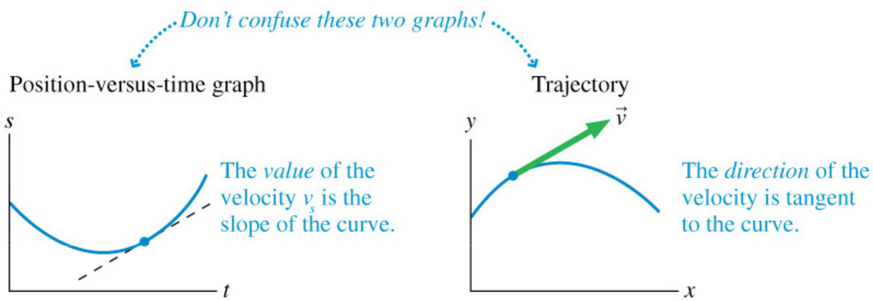


Kinematics in Two Dimensions: Instantaneous Velocity



$$\vec{v} = \left.\frac{\Delta \vec{r}}{\Delta t}\right|_{\Delta t \to 0}$$

Kinematics in Two Dimensions

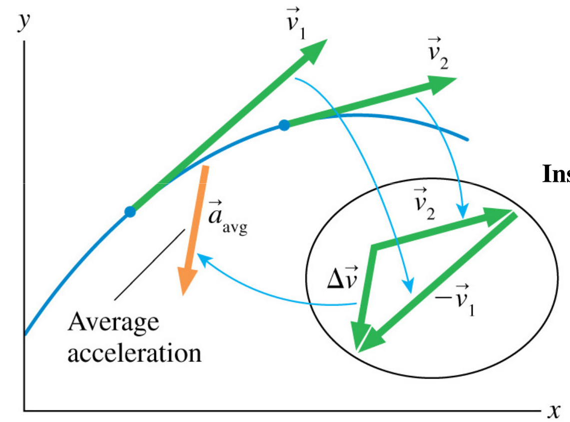


$$v = \frac{ds}{dt}$$

$$\vec{v} = \left.\frac{\Delta \vec{r}}{\Delta t}\right|_{\Delta t \to 0}$$

(motion along a line)

Kinematics in Two Dimensions: Instantaneous Acceleration



Average acceleration:

$$\vec{a}_{avg} = \frac{\Delta \vec{v}}{\Delta t}$$

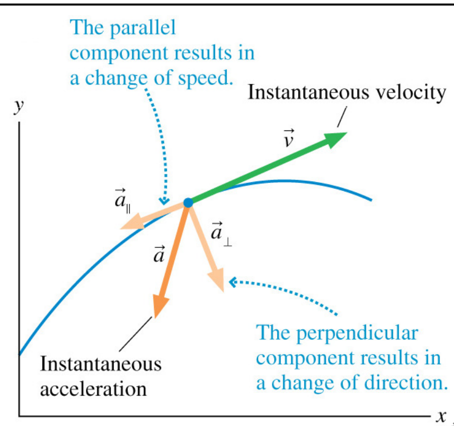
Instantaneous acceleration:

$$\vec{a} = \left.\frac{\Delta \vec{v}}{\Delta t}\right|_{\Delta t \to 0}$$

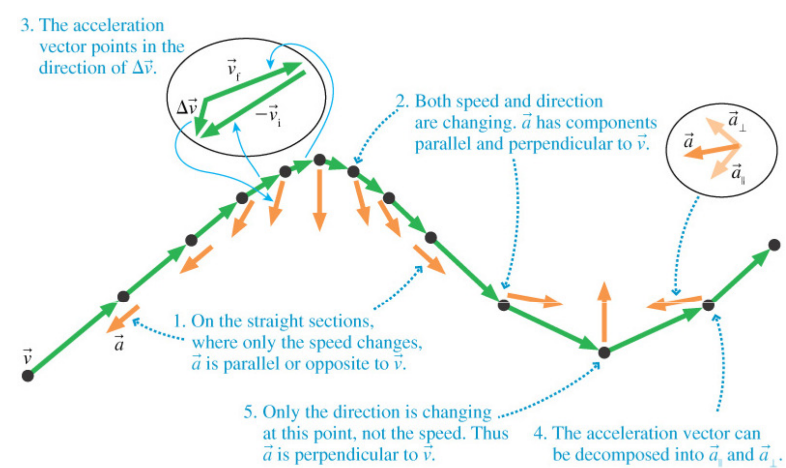
Kinematics in Two Dimensions: Instantaneous Acceleration

<u>Motion along a line</u>: acceleration results in change of speed (the magnitude of velocity)

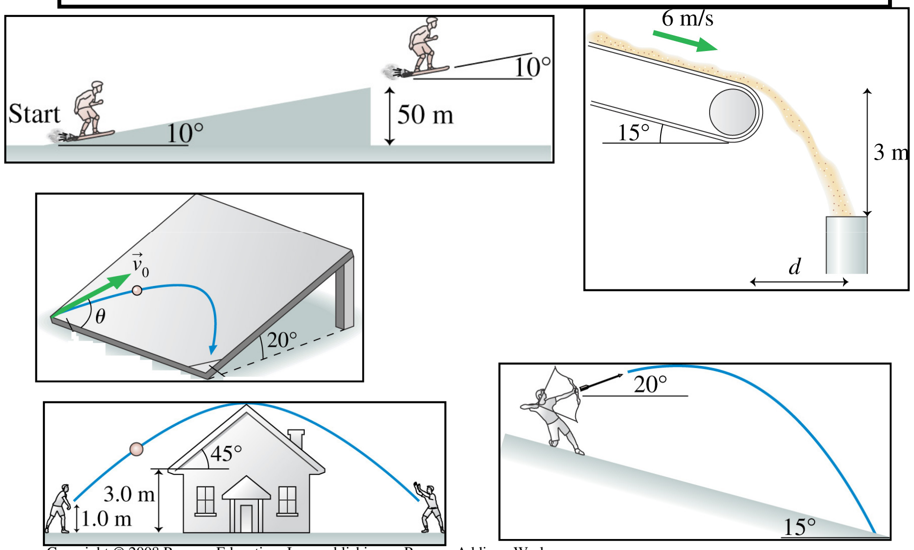
<u>Motion in a plane</u>: acceleration can change the speed (the magnitude of velocity) and the direction of velocity.



$$\vec{a} = \left.\frac{\Delta \vec{v}}{\Delta t}\right|_{\Delta t \to 0}$$



A **Projectile** is an object that moves in two dimensions (in a plane) under the influence of only the **gravitational force.**

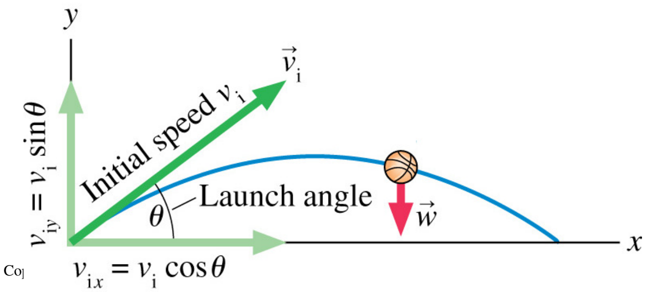


A **Projectile** is an object that moves in two dimensions with free fall acceleration $\vec{a} = \vec{g}$

$a_x = 0$ - motion along x-axis – with zero acceleration – constant velocity

$v_x = \text{constant}$

$a_y = -g = -9.8\frac{m}{s^2}$ - motion along y-axis – free fall motion with constant acceleration



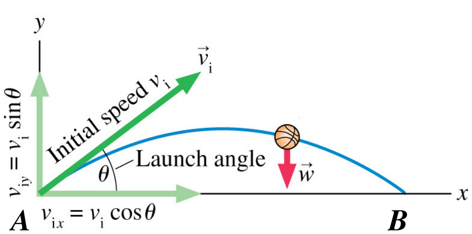
Example: Find the distance AB

$v_x = \text{constant} = v_{i,x} = v_i \cos\theta$ $\qquad x = v_{i,x}t = v_i t \cos\theta$

$a_y = -g = -9.8\frac{m}{s^2}$ $\qquad v_y = v_{i,y} - gt$ $\qquad v_{i,y} = v_i \sin\theta$

Point A - $y_i = y_A = 0$

Point B - $y_{final} = y_B = 0$ **then** $y_{final} = v_{i,y}t_{AB} - g\frac{t_{AB}^2}{2}$ $\qquad v_{i,y}t_{AB} = g\frac{t_{AB}^2}{2}$

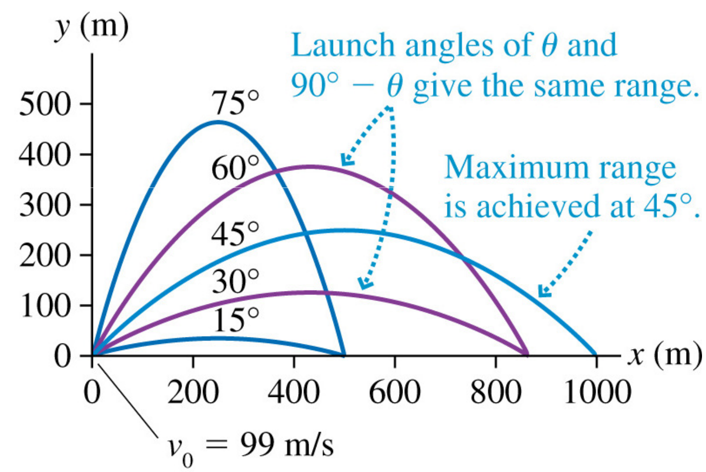


$$t_{AB} = \frac{2v_{i,y}}{g} = \frac{2v_i \sin\theta}{g}$$

then

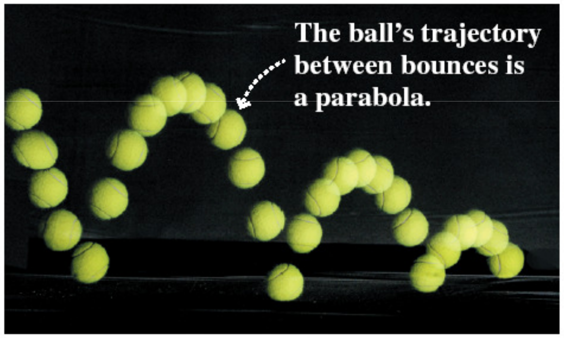
$$x_{AB} = v_i t_{AB} \cos\theta = \frac{2v_i^2}{g}\sin\theta\cos\theta = \frac{v_i^2}{g}\sin 2\theta$$

$$x_{AB} = \frac{v_i^2}{g}\sin 2\theta$$



Example of Projectile Motion

FIGURE 4.15 The parabolic trajectory of a bouncing ball.



EXAMPLE 4.4 Don't try this at home!

QUESTION:

EXAMPLE 4.4 Don't try this at home!
A stunt man drives a car off a 10.0-m-high cliff at a speed of 20.0 m/s. How far does the car land from the base of the cliff?

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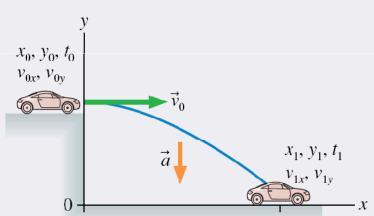
EXAMPLE 4.4 Don't try this at home!

VISUALIZE The pictorial representation, shown in **FIGURE 4.18**, is *very* important because the number of quantities to keep track of in projectile motion problems is quite large. We have chosen to put the origin at the base of the cliff.

FIGURE 4.18 Pictorial representation for the car of Example 4.4.



Known
$x_0 = 0$ m $\quad v_{0y} = 0$ m/s $\quad t_0 = 0$ s
$y_0 = 10.0$ m $\quad v_{0x} = v_0 = 20.0$ m/s
$a_x = 0$ m/s^2 $\quad a_y = -g \quad y_1 = 0$ m

Find
x_1

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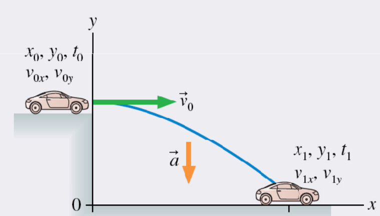
EXAMPLE 4.4 Don't try this at home!

The assumption that the car is moving horizontally as it leaves the cliff leads to $v_{0x} = v_0$ and $v_{0y} = 0$ m/s. A motion diagram is not essential in projectile motion problems because we already know that the projectile follows a parabolic trajectory with $\vec{a} = -g\hat{j}$.

FIGURE 4.18 Pictorial representation for the car of Example 4.4.



Known
$x_0 = 0$ m $\quad v_{0y} = 0$ m/s $\quad t_0 = 0$ s
$y_0 = 10.0$ m $\quad v_{0x} = v_0 = 20.0$ m/s
$a_x = 0$ m/s^2 $\quad a_y = -g \quad y_1 = 0$ m

Find
x_1

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EXAMPLE 4.4 Don't try this at home!

SOLVE Each point on the trajectory has x- and y-components of position, velocity, and acceleration but only *one* value of time. The time needed to move horizontally to x_1 is the *same* time needed to fall vertically through distance y_0. **Although the horizontal and vertical motions are independent, they are connected through the time t.** This is a critical observation for solving projectile motion problems. The kinematics equations are

$$x_1 = x_0 + v_{0x}(t_1 - t_0) = v_0 t_1$$

$$y_1 = 0 = y_0 + v_{0y}(t_1 - t_0) - \tfrac{1}{2}g(t_1 - t_0)^2 = y_0 - \tfrac{1}{2}g t_1^2$$

EXAMPLE 4.4 Don't try this at home!

We can use the vertical equation to determine the time t_1 needed to fall distance y_0:

$$t_1 = \sqrt{\frac{2y_0}{g}} = \sqrt{\frac{2(10.0\text{ m})}{9.80\text{ m/s}^2}} = 1.43\text{ s}$$

We then insert this expression for t into the horizontal equation to find the distance traveled:

$$x_1 = v_0 t_1 = (20.0\text{ m/s})(1.43\text{ s}) = 28.6\text{ m}$$

EXAMPLE 4.4 Don't try this at home!

ASSESS The cliff height is $\approx$33 ft and the initial speed is $v_0 \approx$ 40 mph. Traveling $x_1 = 29$ m $\approx$ 95 ft before hitting the ground seems reasonable.

Problem-Solving Strategy: Projectile Motion Problems

SOLVE The acceleration is known: $a_x = 0$ and $a_y = -g$. Thus the problem is one of two-dimensional kinematics. The kinematic equations are

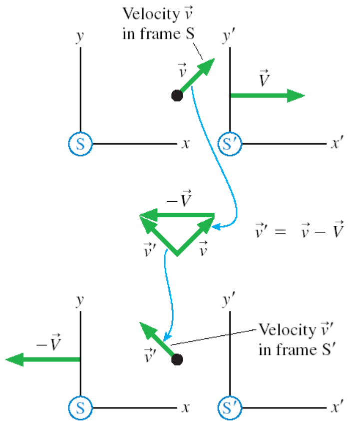
$$x_f = x_i + v_{ix}\,\Delta t \qquad y_f = y_i + v_{iy}\,\Delta t - \tfrac{1}{2}g(\Delta t)^2$$

$$v_{fx} = v_{ix} = \text{constant} \qquad v_{fy} = v_{iy} - g\,\Delta t$$

Δt is the same for the horizontal and vertical components of the motion. Find Δt from one component, then use that value for the other component.

Relative Motion

FIGURE 4.29 Velocities $\vec{v}$ and $\vec{v}'$, as measured in frames S and S′, are related by vector addition.



Relative Motion

If we know an object's velocity measured in one reference frame, S, we can transform it into the velocity that would be measured by an experimenter in a different reference frame, S', using the Galilean transformation of velocity.

$$\vec{v} = \vec{v}' + \vec{V} \qquad \text{or} \qquad \vec{v}' = \vec{v} - \vec{V}$$

Or, in terms of components,

$$\begin{aligned} v_x &= v'_x + V_x \\ v_y &= v'_y + V_y \end{aligned} \qquad \text{or} \qquad \begin{aligned} v'_x &= v_x - V_x \\ v'_y &= v_y - V_y \end{aligned}$$

EXAMPLE 4.8 A speeding bullet

QUESTION:

EXAMPLE 4.8 A speeding bullet
The police are chasing a bank robber. While driving at 50 m/s, they fire a bullet to shoot out a tire of his car. The police gun shoots bullets at 300 m/s. What is the bullet's speed as measured by a TV camera crew parked beside the road?

EXAMPLE 4.8 A speeding bullet

MODEL Assume that all motion is along the x-axis. Let the earth be frame S and a frame attached to the police car be S′. Frame S′ moves relative to frame S with $V_x = 50$ m/s.

EXAMPLE 4.8 A speeding bullet

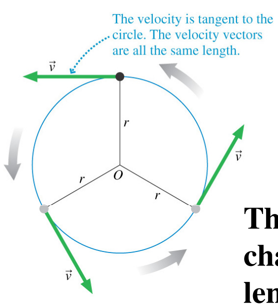
SOLVE The bullet is the moving object that will be observed from both frames. The gun is in frame S′, so the bullet travels in this frame with $v'_x = 300$ m/s. We can use Equation 4.24 to transform the bullet's velocity into the earth reference frame:

$$v_x = v'_x + V_x = 300 \text{ m/s} + 50 \text{ m/s} = 350 \text{ m/s}$$

Motion in a Circle

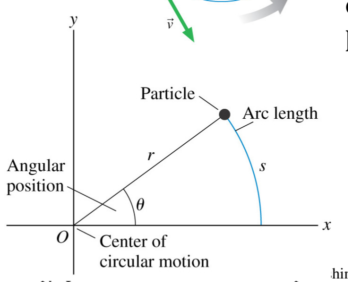
Kinematics in Two Dimensions: Uniform Circular Motion: Motion with Constant Speed

The speed (magnitude of velocity) is the same



Period: $T = \dfrac{2\pi r}{v}$

The position of the particle is characterized by angle (or by arc length)



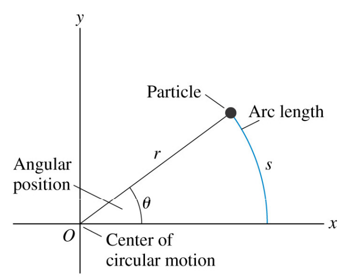
Arc length: $s = r\theta$

where angle θ **is in RADIANS**

$$1\,rad = \frac{360^0}{2\pi} = 57.3^0$$

Example:

What is the arc length for $\theta = 45^0, 90^0, 10^0$ **if** $r = 1m$



$$45^0 = 45^0 \frac{2\pi}{360^0}\,rad = \frac{\pi}{4}\,rad = 0.78\,rad$$

$$90^0 = 90^0 \frac{2\pi}{360^0}\,rad = \frac{\pi}{2}\,rad = 1.57\,rad$$

$$10^0 = 10^0 \frac{2\pi}{360^0}\,rad = \frac{\pi}{18}\,rad = 0.174\,rad$$

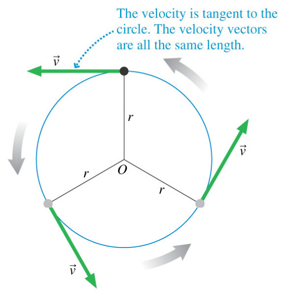
Arc length: $s = r\theta$

$$s_{45} = \frac{\pi r}{4} = 0.78m \qquad s_{10} = \frac{\pi r}{18} = 0.174m$$

$$s_{90} = \frac{\pi r}{2} = 1.57m$$

Kinematics in Two Dimensions: Uniform Circular Motion: Motion with Constant Speed

The speed (magnitude of velocity) is the same



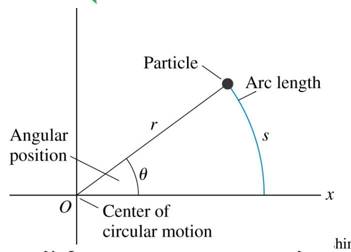
$$s = vt$$

$$\theta = \omega t$$

$$\omega = \frac{v}{r} \qquad T = \frac{2\pi r}{v}$$

Then

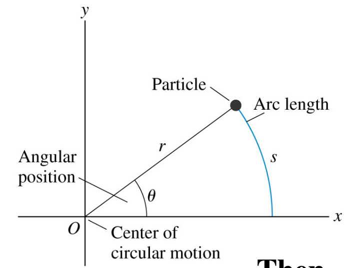
$$\omega = \frac{2\pi}{T}$$



Uniform Circular Motion

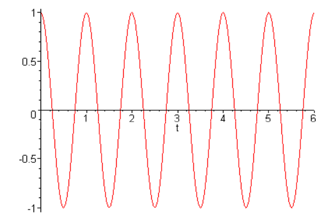
$$s = vt \qquad \omega = \frac{2\pi}{T} \qquad \omega = \frac{v}{r}$$

$$\theta = \omega t$$

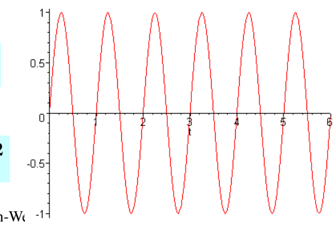


Then

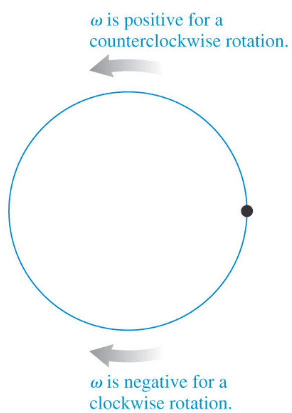
$$x = r\cos\theta = r\cos(\omega t)$$



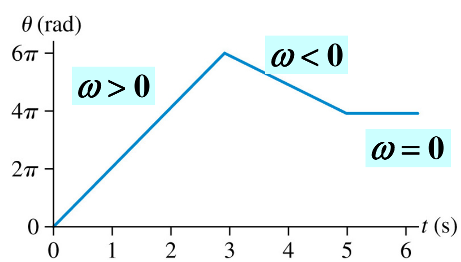
$$y = r\sin\theta = r\sin(\omega t)$$

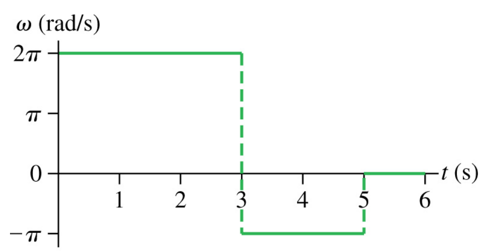


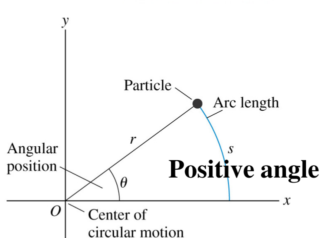
$$x^2 + y^2 = r^2\sin^2(\omega t) + r^2\cos^2(\omega t) = r^2$$



$$\theta = \omega t$$



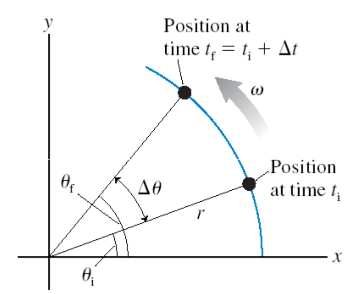




Positive angle

Uniform Circular Motion

FIGURE 4.34 A particle moves with angular velocity ω.



$$\omega \equiv \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad \text{(angular velocity)}$$

$$\theta_f = \theta_i + \omega\,\Delta t \qquad \text{(uniform circular motion)}$$

EXAMPLE 4.13 At the roulette wheel

QUESTIONS:

EXAMPLE 4.13 At the roulette wheel

A small steel roulette ball rolls ccw around the inside of a 30-cm-diameter roulette wheel. The ball completes 2.0 rev in 1.20 s.

a. What is the ball's angular velocity?
b. What is the ball's position at $t = 2.0$ s? Assume $\theta_i = 0$.

EXAMPLE 4.13 At the roulette wheel

SOLVE a. The period of the ball's motion, the time for 1 rev, is $T = 0.60$ s. Angular velocity is positive for ccw motion, so

$$\omega = \frac{2\pi \text{ rad}}{T} = \frac{2\pi \text{ rad}}{0.60 \text{ s}} = 10.47 \text{ rad/s}$$

EXAMPLE 4.13 At the roulette wheel

b. The ball starts at $\theta_i = 0$ rad. After $\Delta t = 2.0$ s, its position is given by Equation 4.30:

$$\theta_f = 0 \text{ rad} + (10.47 \text{ rad/s})(2.0 \text{ s}) = 20.94 \text{ rad}$$

where we've kept an extra significant figure to avoid round-off error. Although this is a mathematically acceptable answer, an observer would say that the ball is always located somewhere between 0° and 360°. Thus it is common practice to subtract an integer number of 2π rad, representing the completed revolutions.

EXAMPLE 4.13 At the roulette wheel

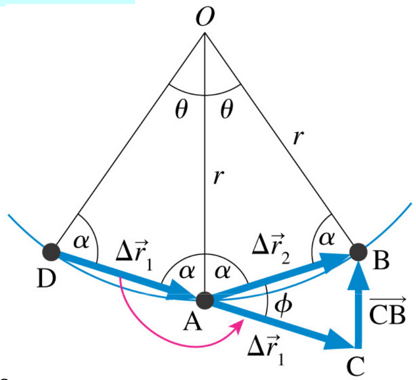
Because $20.94/2\pi = 3.333$, we can write

$$\begin{aligned}\theta_f = 20.94 \text{ rad} &= 3.333 \times 2\pi \text{ rad} \\ &= 3 \times 2\pi \text{ rad} + 0.333 \times 2\pi \text{ rad} \\ &= 3 \times 2\pi \text{ rad} + 2.09 \text{ rad}\end{aligned}$$

In other words, at $t = 2.0$ s the ball has completed 3 rev and is 2.09 rad = 120° into its fourth revolution. An observer would say that the ball's position is $\theta_f = 120°$.

Uniform Circular Motion: Acceleration

$$\vec{a} = \frac{\Delta\vec{v}}{\Delta t}\bigg|_{\Delta t\to 0} \qquad \vec{v} = \frac{\Delta\vec{r}}{\Delta t}\bigg|_{\Delta t\to 0}$$



$$\Delta\vec{r}_1 = \vec{v}_1\Delta t$$

$$\Delta\vec{r}_2 = \vec{v}_2\Delta t$$

$$\vec{a} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\Delta\vec{r}_2 - \Delta\vec{r}_1}{(\Delta t)^2}$$

Direction of acceleration is the same as direction of vector (toward the center) $\overrightarrow{CB} = \Delta\vec{r}_2 - \Delta\vec{r}_1$

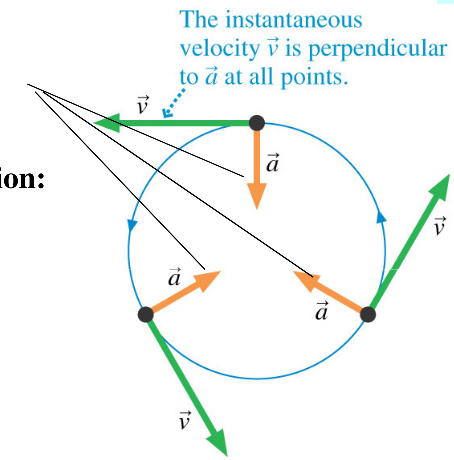
The magnitude of acceleration:

$$a = \frac{CB}{(\Delta t)^2} = \frac{\Delta r\phi}{(\Delta t)^2} = \frac{\Delta r(\pi - 2\alpha)}{(\Delta t)^2} = \frac{\Delta r\theta}{(\Delta t)^2} = \frac{v\Delta t\,\omega\Delta t}{(\Delta t)^2} = v\omega = \frac{v^2}{r}$$

Uniform Circular Motion: Acceleration

$$\omega = \frac{v}{r}$$

centripetal acceleration



The magnitude of acceleration:

$$a = \frac{v^2}{r} = \omega^2 r = v\omega$$

$$\vec{a} = \left(\frac{v^2}{r}, \text{ toward center of circle}\right) \qquad \text{(centripetal acceleration)}$$

EXAMPLE 4.14 The acceleration of a Ferris wheel

QUESTION:

EXAMPLE 4.14 The acceleration of a Ferris wheel

A typical carnival Ferris wheel has a radius of 9.0 m and rotates 4.0 times per minute. What magnitude acceleration do the riders experience?

EXAMPLE 4.14 The acceleration of a Ferris wheel

MODEL Model the rider as a particle in uniform circular motion.

EXAMPLE 4.14 The acceleration of a Ferris wheel

SOLVE The period is $T = \frac{1}{4}\text{ min} = 15\text{ s}$. From Equation 4.25, a rider's speed is

$$v = \frac{2\pi r}{T} = \frac{2\pi(9.0\text{ m})}{15\text{ s}} = 3.77\text{ m/s}$$

Consequently, the centripetal acceleration is

$$a = \frac{v^2}{r} = \frac{(3.77\text{ m/s})^2}{9.0\text{ m}} = 1.6\text{ m/s}^2$$

Chapter 4. Summary Slides

General Principles

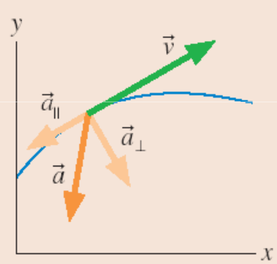
The instantaneous velocity

$$\vec{v} = d\vec{r}/dt$$

is a vector tangent to the trajectory.

The instantaneous acceleration is

$$\vec{a} = d\vec{v}/dt$$



$\vec{a}_{\parallel}$, the component of $\vec{a}$ parallel to $\vec{v}$, is responsible for change of *speed*. $\vec{a}_{\perp}$, the component of $\vec{a}$ perpendicular to $\vec{v}$, is responsible for change of *direction*.

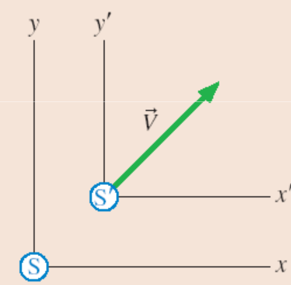
General Principles

Relative motion

Inertial reference frames move relative to each other with constant velocity $\vec{V}$. Measurements of position and velocity measured in frame S are related to measurements in frame S′ by the Galilean transformations:

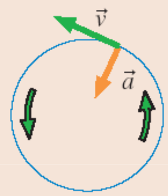
$$x' = x - V_x t \qquad v'_x = v_x - V_x$$

$$y' = y - V_y t \qquad v'_y = v_y - V_y$$



Important Concepts

Uniform Circular Motion



Angular velocity $\omega = d\theta/dt$.
v_t and ω are constant:

$$v_t = \omega r$$

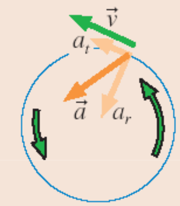
The centripetal acceleration points toward the center of the circle:

$$a = \frac{v^2}{r} = \omega^2 r$$

It changes the particle's direction but not its speed.

Important Concepts

Nonuniform Circular Motion



Angular acceleration $\alpha = d\omega/dt$.
The radial acceleration

$$a_r = \frac{v^2}{r} = \omega^2 r$$

changes the particle's direction. The tangential component

$$a_t = \alpha r$$

changes the particle's speed.

Applications

Kinematics in two dimensions

If $\vec{a}$ is constant, then the x- and y-components of motion are independent of each other.

$$x_f = x_i + v_{ix}\Delta t + \tfrac{1}{2}a_x(\Delta t)^2$$

$$y_f = y_i + v_{iy}\Delta t + \tfrac{1}{2}a_y(\Delta t)^2$$

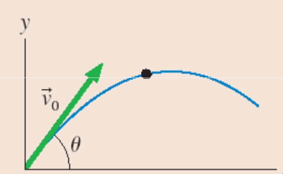
$$v_{fx} = v_{ix} + a_x\Delta t$$

$$v_{fy} = v_{iy} + a_y\Delta t$$

Applications

Projectile motion occurs if the object moves under the influence of only gravity. The motion is a parabola.

- Uniform motion in the horizontal direction with $v_{0x} = v_0\cos\theta$.
- Free-fall motion in the vertical direction with $a_y = -g$ and $v_{0y} = v_0\sin\theta$.
- The x and y kinematic equations have the *same* value for Δt.

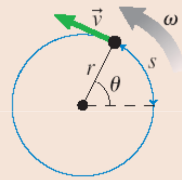


Applications

Circular motion kinematics

Period $T = \dfrac{2\pi r}{v} = \dfrac{2\pi}{\omega}$

Angular position $\theta = \dfrac{s}{r}$



$$\omega_f = \omega_i + \alpha\Delta t$$

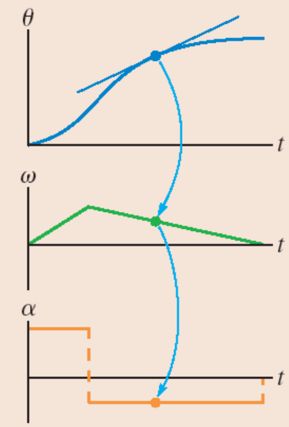
$$\theta_f = \theta_i + \omega_i\Delta t + \tfrac{1}{2}\alpha(\Delta t)^2$$

$$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$$

Applications

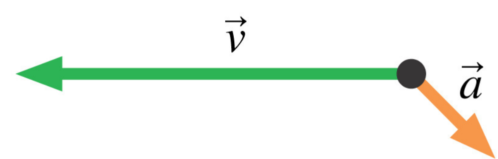
Angle, angular velocity, and angular acceleration are related graphically.

- The angular velocity is the slope of the angular position graph.
- The angular acceleration is the slope of the angular velocity graph.



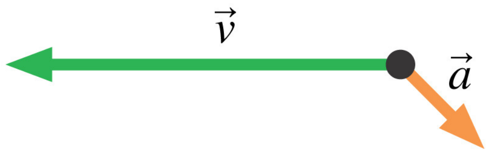
Chapter 4. Questions

This acceleration will cause the particle to



A. slow down and curve downward.
B. slow down and curve upward.
C. speed up and curve downward.
D. speed up and curve upward.
E. move to the right and down.

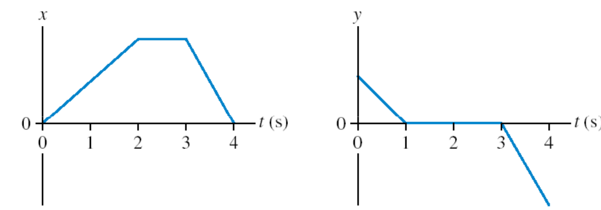
This acceleration will cause the particle to



A. slow down and curve downward.
B. slow down and curve upward.
C. speed up and curve downward.
D. speed up and curve upward.
E. move to the right and down.

During which time interval is the particle described by these position graphs at rest?

A. 0–1 s
B. 1–2 s
C. 2–3 s
D. 3–4 s



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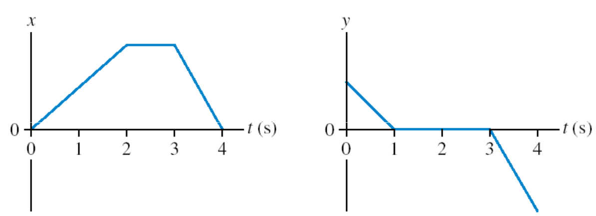

A 50 g ball rolls off a table and lands 2 m from the base of the table. A 100 g ball rolls off the same table with the same speed. It lands at a distance

A. less than 2 m from the base.
B. 2 m from the base.
C. greater than 2 m from the base.

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A plane traveling horizontally to the right at 100 m/s flies past a helicopter that is going straight up at 20 m/s. From the helicopter's perspective, the plane's direction and speed are

A. right and up, more than 100 m/s.
B. right and up, less than 100 m/s.
C. right and down, more than 100 m/s.
D. right and down, less than 100 m/s.
E. right and down, 100 m/s.

A plane traveling horizontally to the right at 100 m/s flies past a helicopter that is going straight up at 20 m/s. From the helicopter's perspective, the plane's direction and speed are

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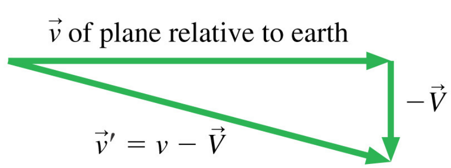
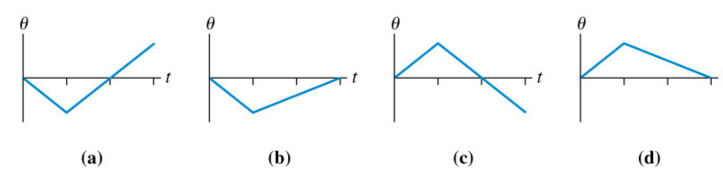


A particle moves cw around a circle at constant speed for 2.0 s. It then reverses direction and moves ccw at half the original speed until it has traveled through the same angle. Which is the particle's angle-versus-time graph?



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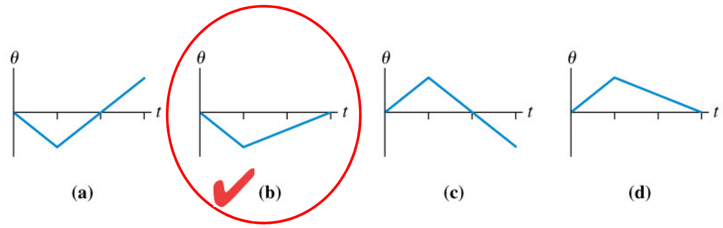
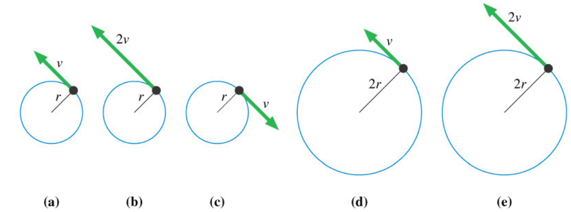


Rank in order, from largest to smallest, the centripetal accelerations $(a_r)_a$ to $(a_r)_e$ of particles a to e.



A. $(a_r)_b > (a_r)_e > (a_r)_a > (a_r)_d > (a_r)_c$
B. $(a_r)_b > (a_r)_e > (a_r)_a = (a_r)_c > (a_r)_d$
C. $(a_r)_b = (a_r)_e > (a_r)_a = (a_r)_c > (a_r)_d$
D. $(a_r)_b > (a_r)_a = (a_r)_c = (a_r)_e > (a_r)_d$
E. $(a_r)_b > (a_r)_a = (a_r)_a > (a_r)_e > (a_r)_d$

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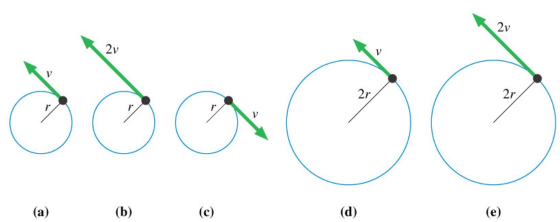

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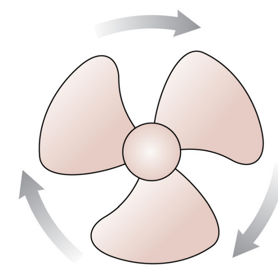
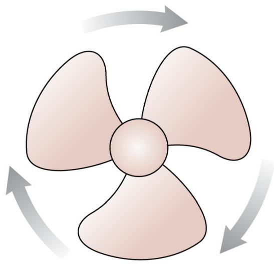
The fan blade is slowing down. What are the signs of ω and α?



A. ω is positive and α is positive.
B. ω is negative and α is positive.
C. ω is positive and α is negative.
D. ω is negative and α is negative.

The fan blade is slowing down. What are the signs of ω and α?



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