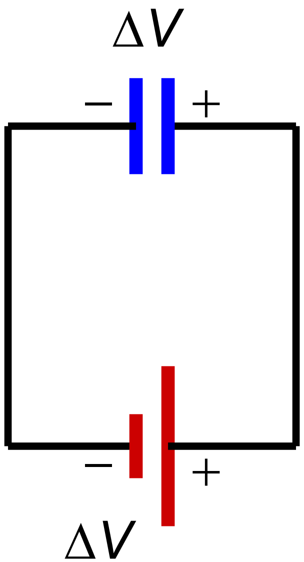
Chapter 33

Alternating Current Circuits

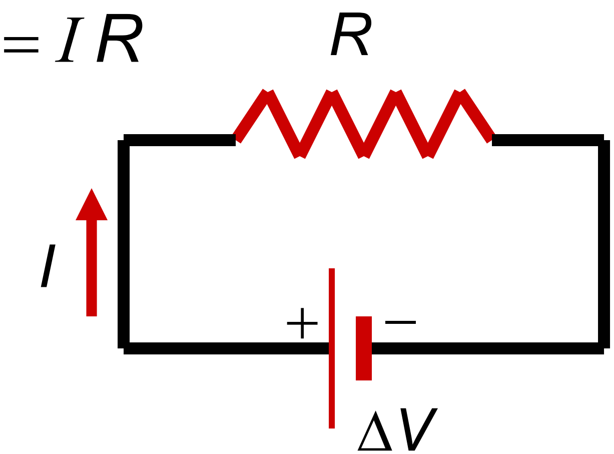
Capacitor



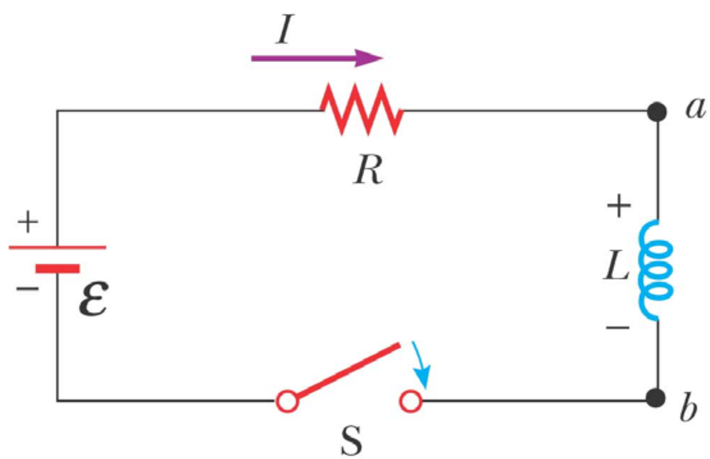
$$Q = C\Delta V$$

Resistor

$$\Delta V = I R$$



Inductance



$$\Delta V_{ab} = -L\frac{dI}{dt}$$

AC power source

- The AC power source provides an alternative voltage, $\Delta v(t)$
- Notation
  - Lower case symbols will indicate instantaneous values
  - Capital letters will indicate fixed values

- The output of an AC power source is sinusoidal

$$\Delta v = \Delta V_{max} \sin \omega t$$

  - Δv is the instantaneous voltage
  - ΔV_{max} is the maximum output voltage of the source
  - ω is the angular frequency of the AC voltage

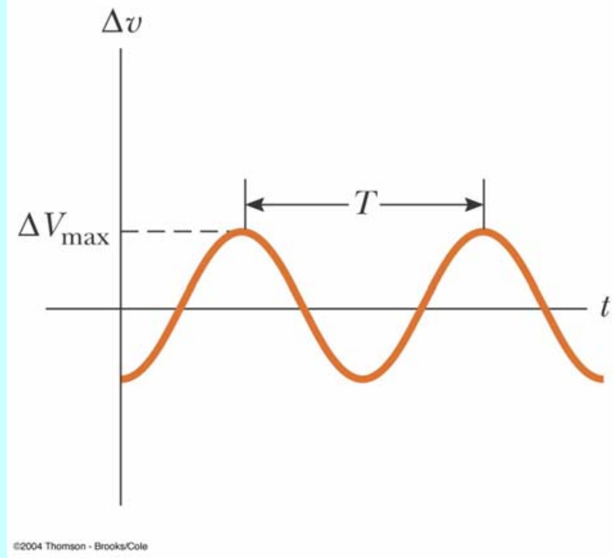
AC voltage

$$\Delta v = \Delta V_{max} \cos(\omega t + \varphi)$$

- The angular frequency is

$$\omega = 2\pi f = \frac{2\pi}{T}$$

  - f is the frequency of the source
  - T is the period of the source
- The voltage is positive during one half of the cycle and negative during the other half
- The current in any circuit driven by an AC source is an alternating current that varies sinusoidally with time
- Commercial electric power plants in the US use a frequency of 60 Hz

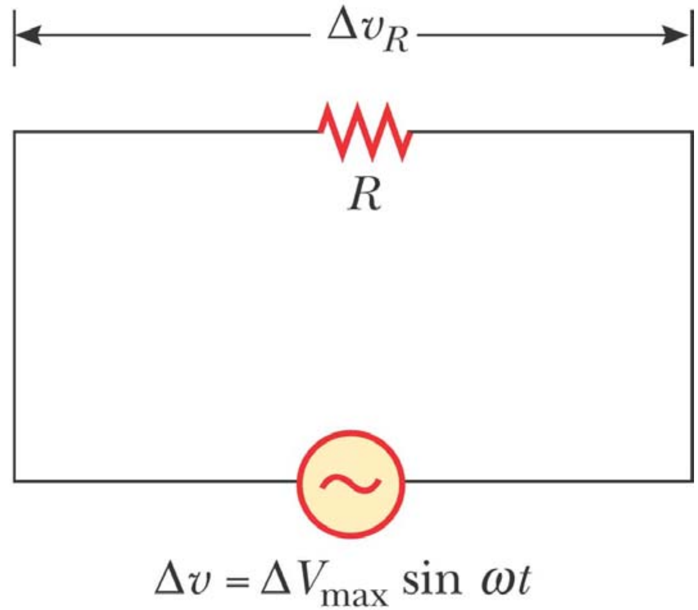


Resistor in AC circuit

- Consider a circuit consisting of an AC source and a resistor
- The AC source is symbolized by

- $\Delta v = \Delta v_R = \Delta V_{max} \sin \omega t$
- Δv_R is the instantaneous voltage across the resistor
- The instantaneous current in the resistor is

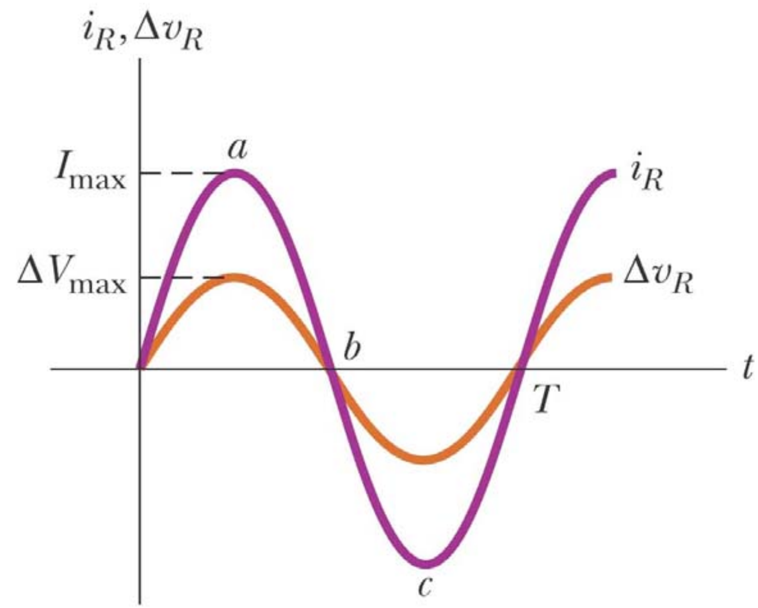
$$i_R = \frac{\Delta v_R}{R} = \frac{\Delta V_{max}}{R} \sin \omega t = I_{max} \sin \omega t$$



Resistor in AC circuit

$$\Delta v = \Delta v_R = \Delta V_{max} \sin(\omega t)$$

$$i_R = \frac{\Delta v_R}{R} = \frac{\Delta V_{max}}{R} \sin \omega t = I_{max} \sin \omega t$$



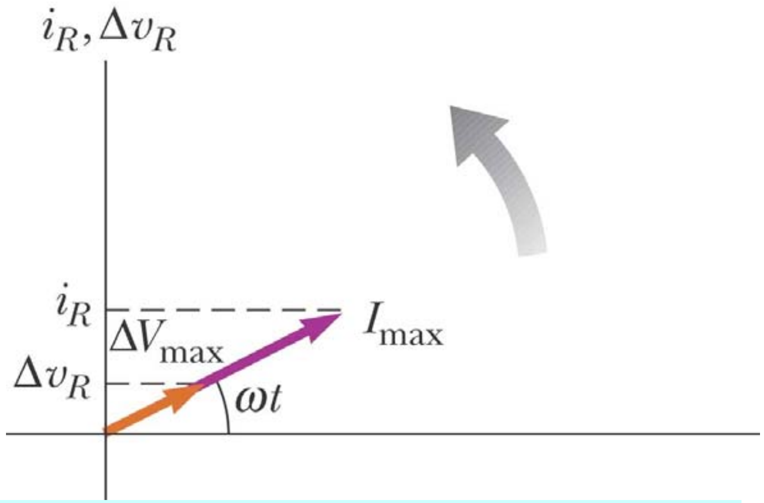
(a)

- The current and the voltage are ***in phase***
- Resistors behave essentially the same way in both DC and AC circuits

Resistor in AC circuit: Phasor diagram

$$\Delta v = \Delta v_R = \Delta V_{max} \sin(\omega t)$$

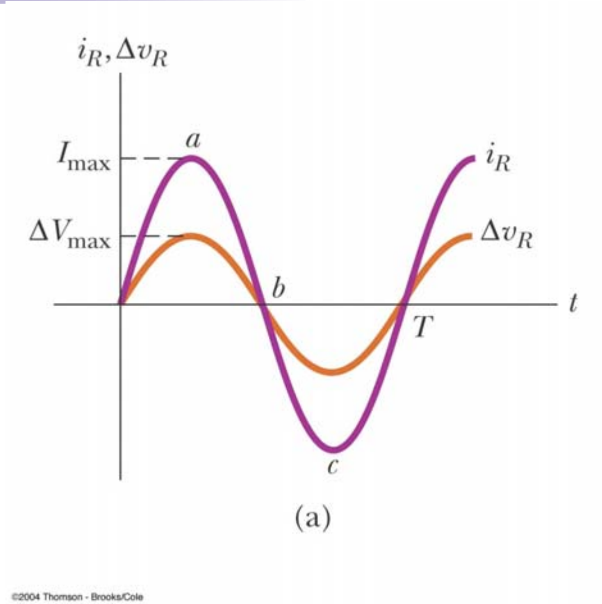
$$i_R = \frac{\Delta v_R}{R} = \frac{\Delta V_{max}}{R} \sin \omega t = I_{max} \sin \omega t$$



- A **phasor** is a vector whose length is proportional to the maximum value of the variable it represents
- The vector rotates at an angular speed equal to the angular frequency associated with the variable
- The projection of the phasor onto the vertical axis represents the instantaneous value of the quantity it represents

rms current and voltage

$$i_R = I_{max} \sin \omega t$$



- The average current in one cycle is zero
- ***rms*** stands for *root mean square*

$$I_{rms} = \left(\frac{1}{T} \int_0^T i_R^2 \, dt \right)^{1/2} = \left(\frac{1}{T} I_{max}^2 \int_0^T \sin^2(\omega t) dt \right)^{1/2}$$

$$= \left(\frac{1}{2\pi} I_{max}^2 \int_0^{2\pi} \sin^2(\tau) d\tau \right)^{1/2} = \frac{I_{max}}{\sqrt{2}} = 0.707\, I_{max}$$

- Alternating voltages can also be discussed in terms of rms values

$$\Delta V_{rms} = \frac{\Delta V_{max}}{\sqrt{2}} = 0.707\, \Delta V_{max}$$

rms current and voltage: power

- The rate at which electrical energy is dissipated in the circuit is given by

$$P = i^2 R$$

    - where i is the *instantaneous current*

- The average power delivered to a resistor that carries an alternating current is

$$P_{av} = I_{rms}^2 R$$

Inductors in AC circuit

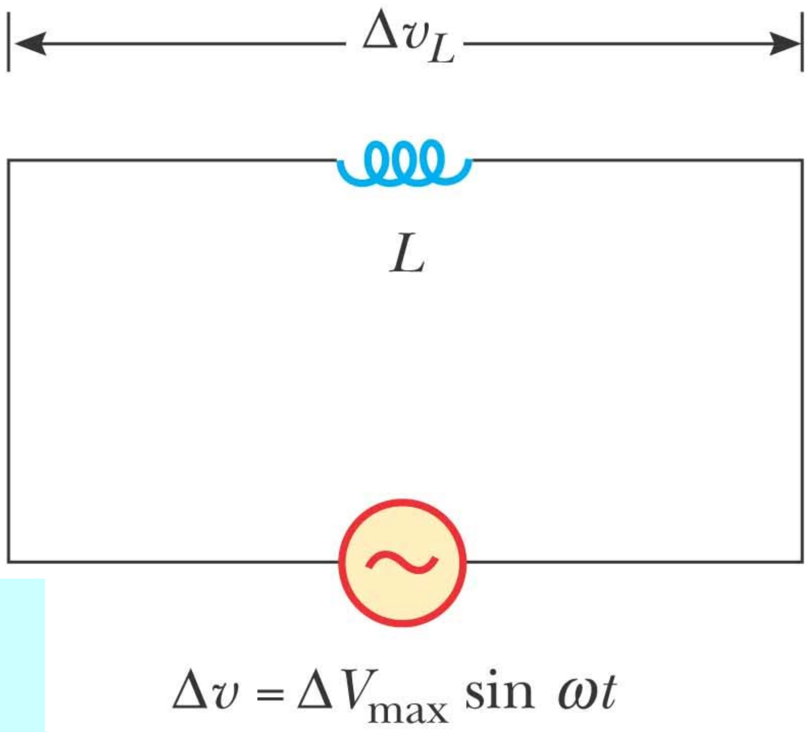
$$\Delta v + \Delta v_L = 0, \text{ or}$$

$$\Delta v - L\frac{di}{dt} = 0$$

$$\Delta v = L\frac{di}{dt} = \Delta V_{max}\, sin\, \omega t$$

$$i_L = \frac{\Delta V_{max}}{L}\int \sin\, \omega t\; dt = -\frac{\Delta V_{max}}{\omega L}cos\, \omega t$$

$$i_L = \frac{\Delta V_{max}}{\omega L}sin\left(\omega t - \frac{\pi}{2}\right) \qquad I_{max} = \frac{\Delta V_{max}}{\omega L}$$



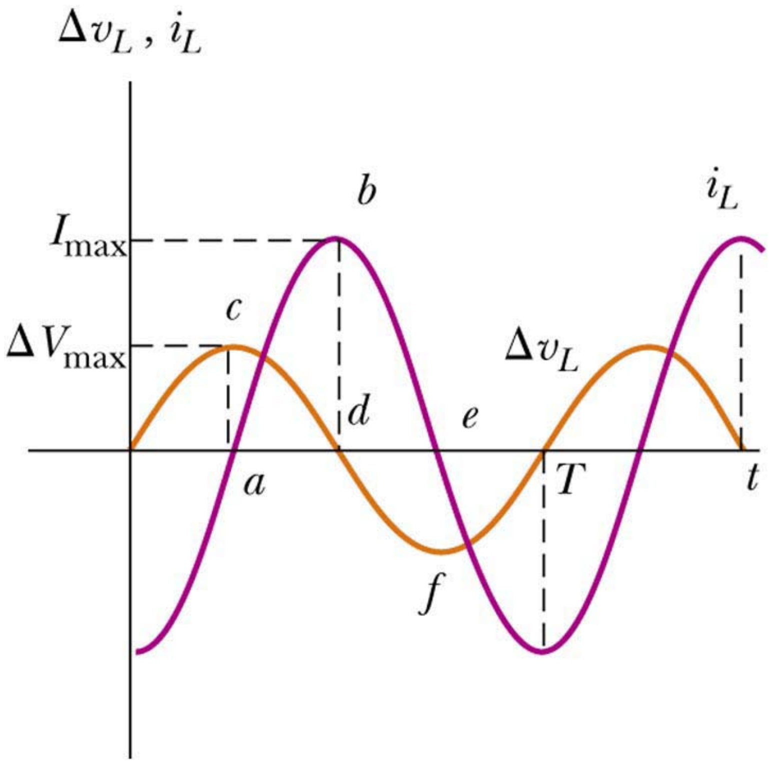
This shows that the instantaneous current i_L in the inductor and the instantaneous voltage Δv_L across the inductor are ***out of phase*** by $(\pi/2)$ rad = 90°.

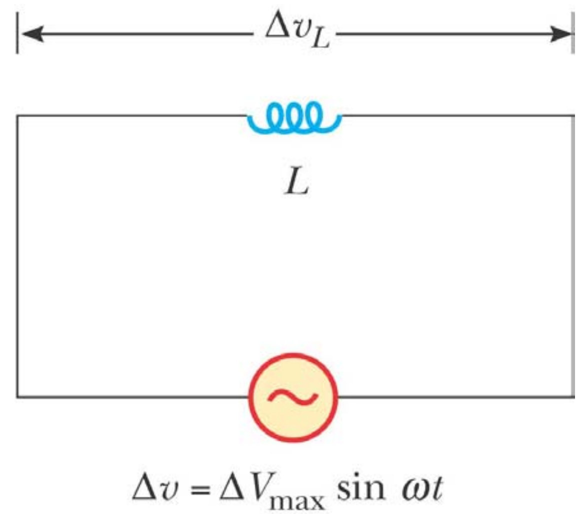
Inductors in AC circuit

$$\Delta v = \Delta V_{max} \sin \omega t$$

$$i_L = I_{max} \sin\left(\omega t - \frac{\pi}{2}\right)$$

$$I_{max} = \frac{\Delta V_{max}}{\omega L}$$

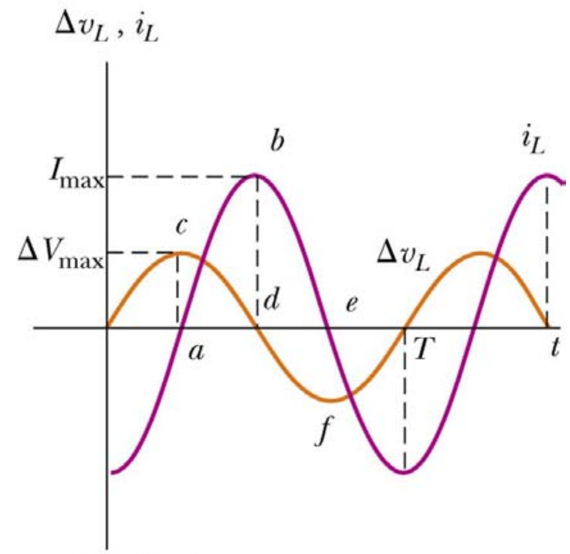




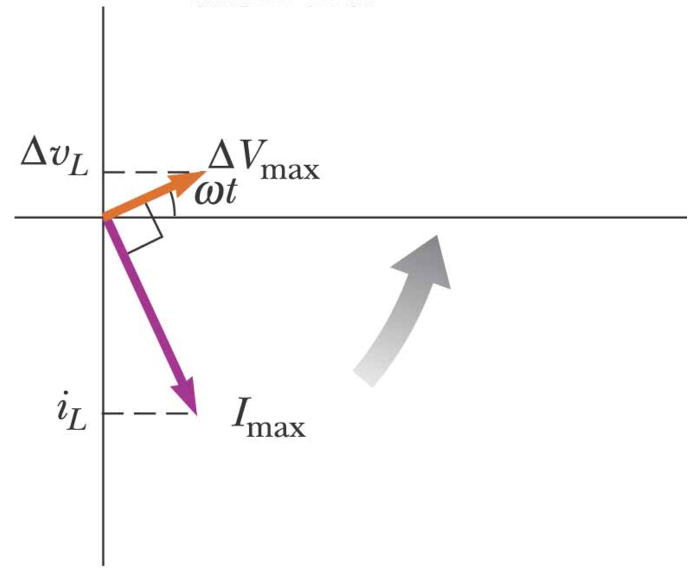
Inductors in AC circuit

$$\Delta v = \Delta V_{max} \sin \omega t$$

$$i_L = I_{max} \sin\left(\omega t - \frac{\pi}{2}\right) \qquad I_{max} = \frac{\Delta V_{max}}{\omega L}$$



- The phasors are at 90º with respect to each other
- This represents the phase difference between the current and voltage
- Specifically, the current lags behind the voltage by 90º

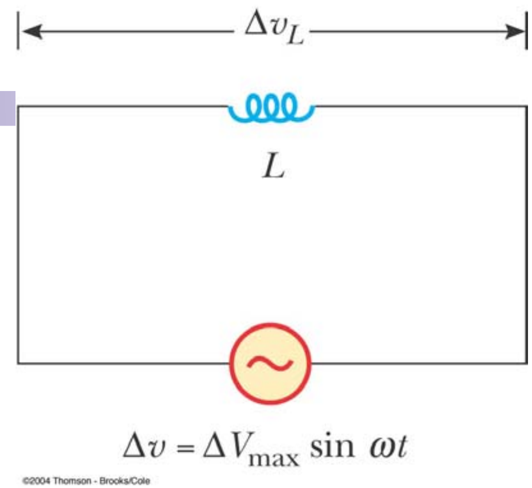


(b)

Inductors in AC circuit

$$\Delta v = \Delta V_{max} \sin \omega t$$

$$i_L = I_{max} \sin\left(\omega t - \frac{\pi}{2}\right) \qquad I_{max} = \frac{\Delta V_{max}}{\omega L}$$



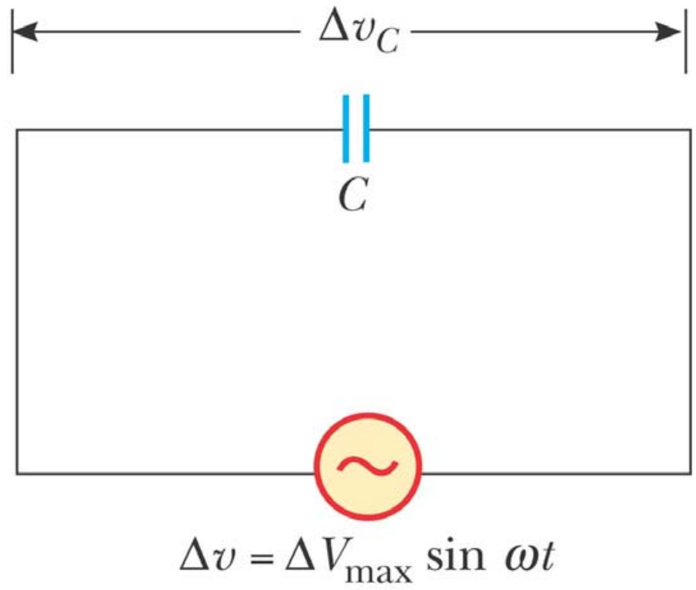
- The factor ωL has the same units as resistance and is related to current and voltage in the same way as resistance
- The factor is the **inductive reactance** and is given by:

$$X_L = \omega L$$

  – As the frequency increases, the inductive reactance increases

$$I_{max} = \frac{\Delta V_{max}}{X_L}$$

Capacitors in AC circuit



$\Delta v + \Delta v_c = 0$ and so

$\Delta v = \Delta v_C = \Delta V_{max} \sin \omega t$

– Δv_c is the instantaneous voltage across the capacitor

- The charge is

$$q = C\Delta v_C = C\Delta V_{max} \sin \omega t$$

- The instantaneous current is given by

$$i_C = \frac{dq}{dt} = \omega C \Delta V_{max} \cos \omega t$$

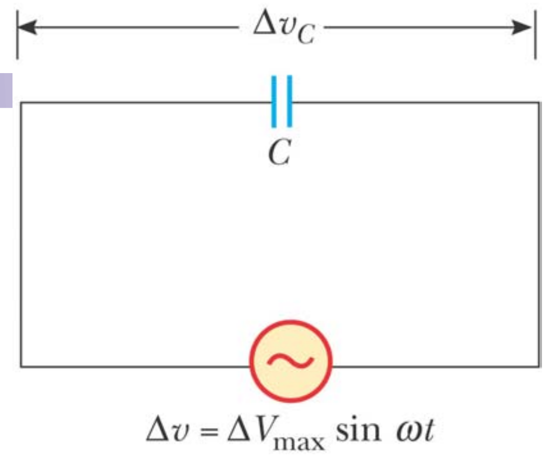
$$i_C = \omega C \Delta V_{max} \sin\left(\omega t + \frac{\pi}{2}\right)$$

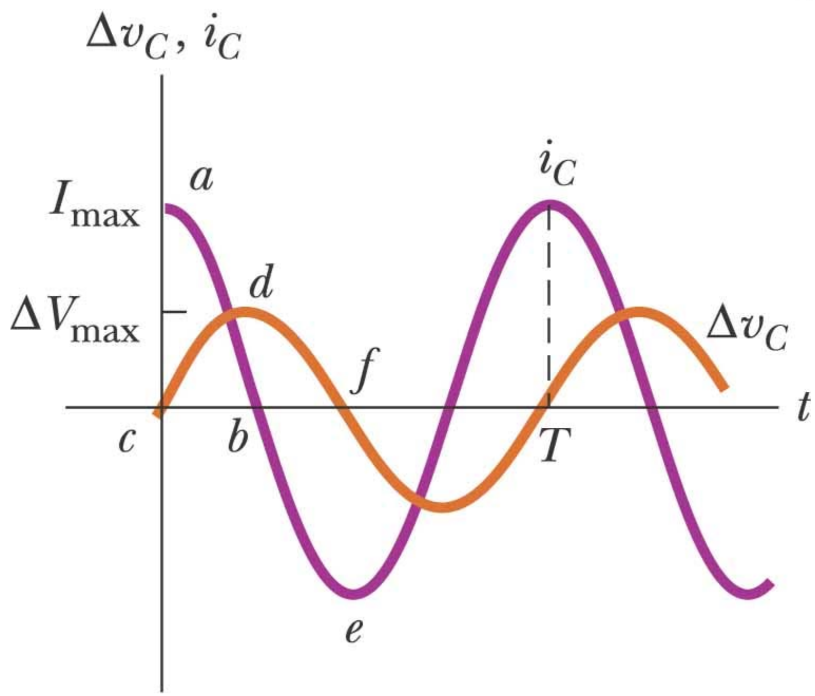
- The current is $(\pi/2)$ rad = 90° out of phase with the voltage

Capacitors in AC circuit

$$\Delta v_C = \Delta V_{max} \sin \omega t$$

$$i_C = \omega C \Delta V_{max} \sin\left(\omega t + \frac{\pi}{2}\right)$$



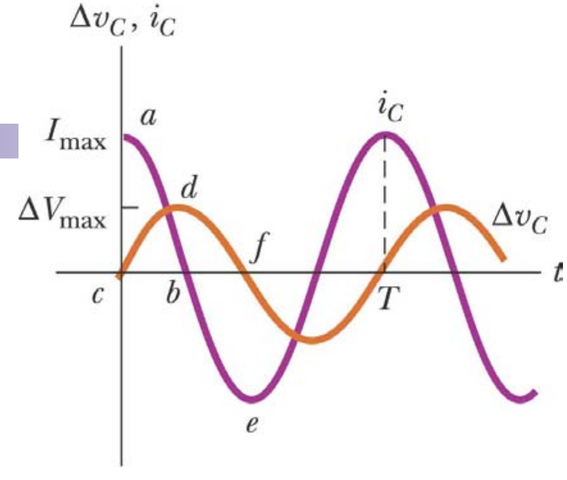


(a)

Capacitors in AC circuit

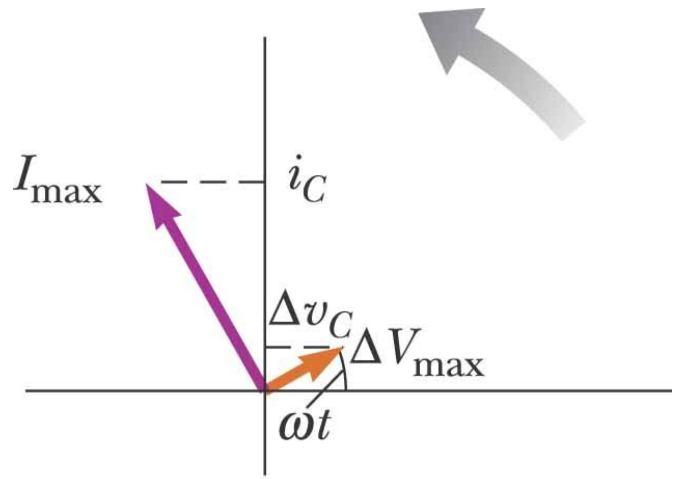
$$\Delta v_C = \Delta V_{max} \sin \omega t$$

$$i_C = \omega C \Delta V_{max} \sin\left(\omega t + \frac{\pi}{2}\right)$$



(a)

- The phasor diagram shows that for a sinusoidally applied voltage, the current always leads the voltage across a capacitor by 90°
  - This is equivalent to saying the voltage lags the current

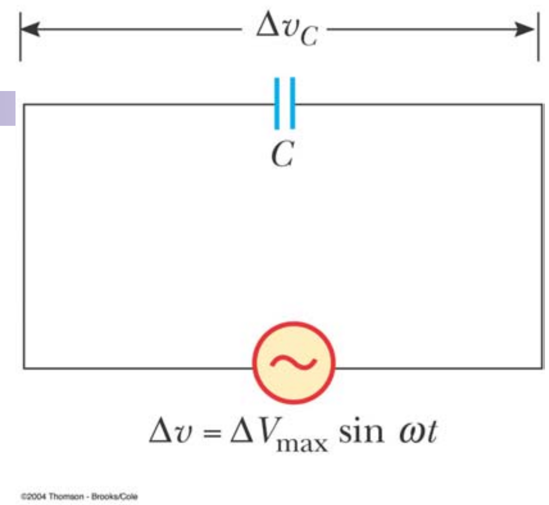


(b)

Capacitors in AC circuit

$$\Delta v_C = \Delta V_{max} \sin \omega t$$

$$i_C = \omega C \Delta V_{\max} \sin\left(\omega t + \frac{\pi}{2}\right)$$

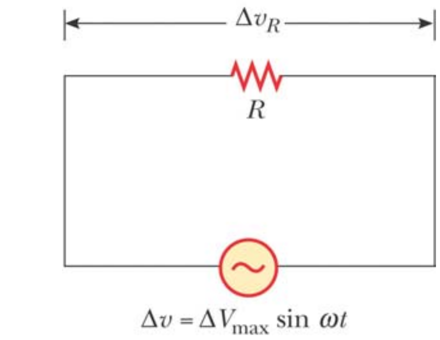


- The maximum current

$$I_{\max} = \omega C \Delta V_{\max} = \frac{\Delta V_{\max}}{(1/\omega C)}$$

- The impeding effect of a capacitor on the current in an AC circuit is called the **capacitive reactance** and is given by

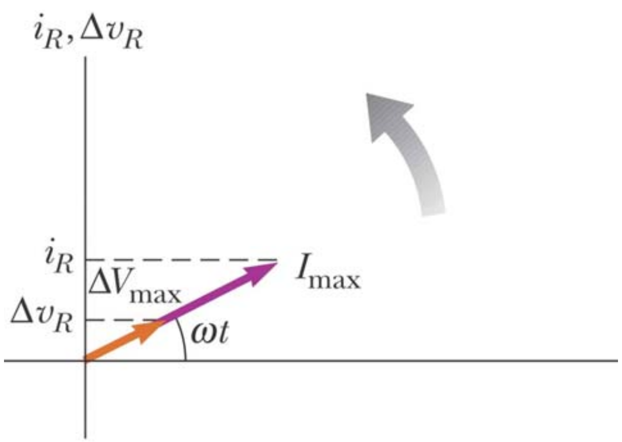
$$X_C \equiv \frac{1}{\omega C} \quad \text{and} \quad I_{\max} = \frac{\Delta V_{\max}}{X_C}$$

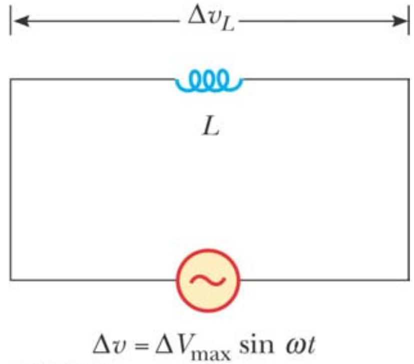


$$\Delta v = \Delta V_{max} \sin\omega t$$

$$i_L = I_{max} \sin\omega t$$

$$I_{max} = \frac{\Delta V_{max}}{R}$$

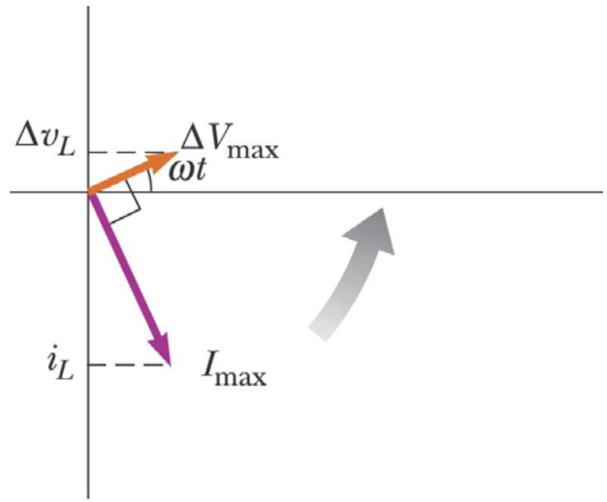




$$\Delta v = \Delta V_{max} \sin\omega t$$

$$i_L = I_{max} \sin\left(\omega t - \frac{\pi}{2}\right)$$

$$I_{max} = \frac{\Delta V_{max}}{\omega L} = \frac{\Delta V_{max}}{X_L}$$



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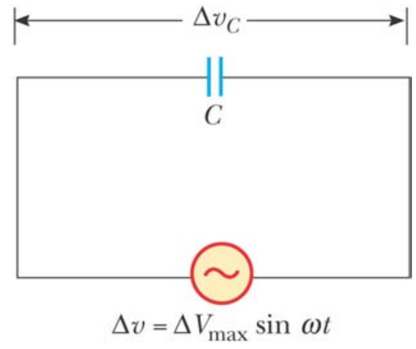
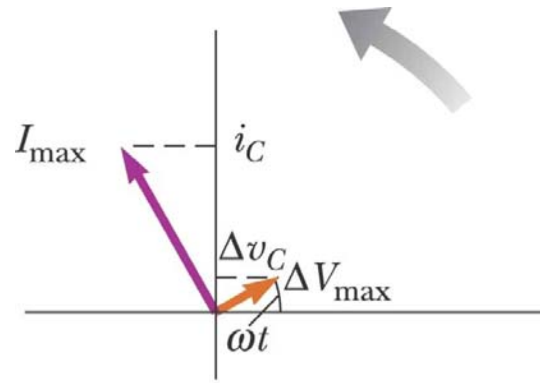


$$\Delta v_C = \Delta V_{max} \sin\omega t$$

$$i_C = I_{\max} \sin\left(\omega t + \frac{\pi}{2}\right)$$

$$I_{\max} = \frac{\Delta V_{\max}}{(1/\omega C)} = \frac{\Delta V_{\max}}{X_C}$$



(b)

RLC series circuit

- The instantaneous voltage would be given by

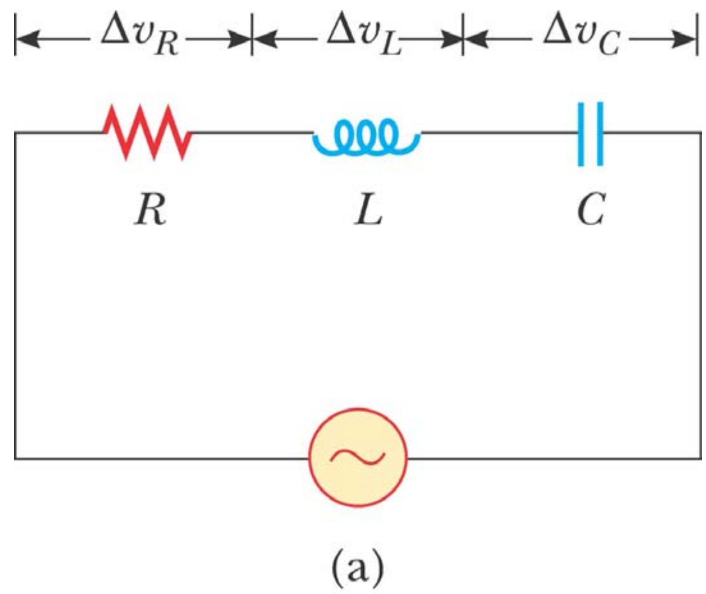
$$\Delta v = \Delta V_{max} \sin \omega t$$

- The instantaneous current would be given by

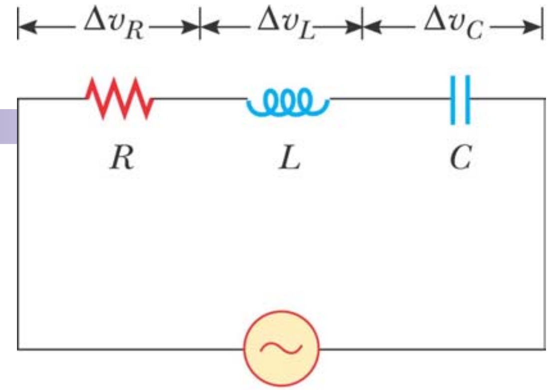
$$i = I_{max} \sin (\omega t - \varphi)$$

  - φ is the *phase angle* between the current and the applied voltage

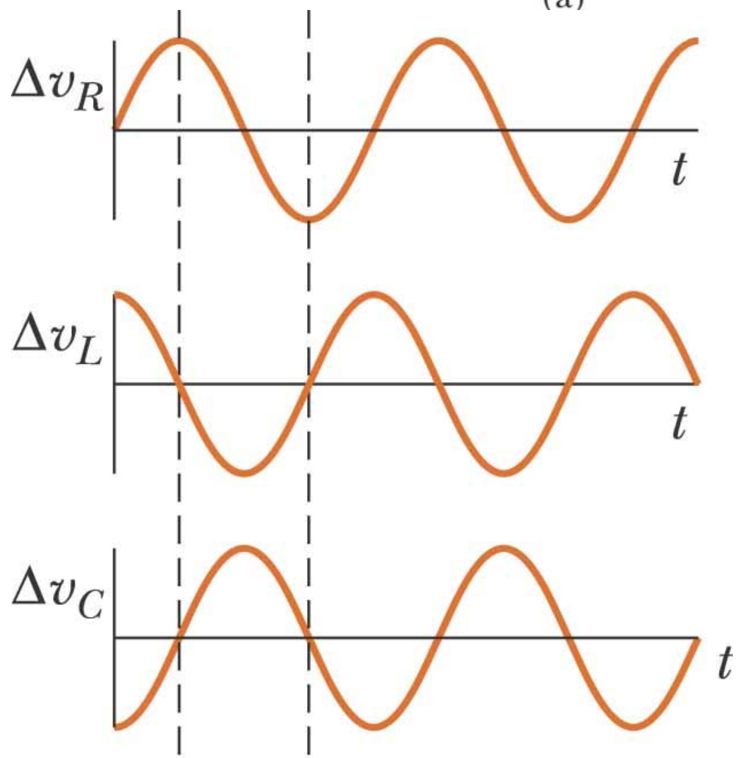
- Since the elements are in series, the current at all points in the circuit has the same amplitude and phase



RLC series circuit



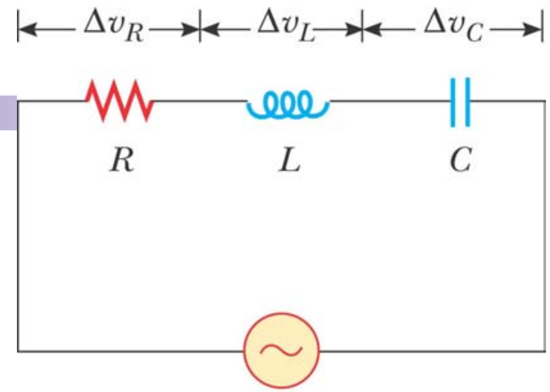
(a)



(b)

- The instantaneous voltage across the resistor is in phase with the current
- The instantaneous voltage across the inductor leads the current by 90°
- The instantaneous voltage across the capacitor lags the current by 90°

RLC series circuit



(a)

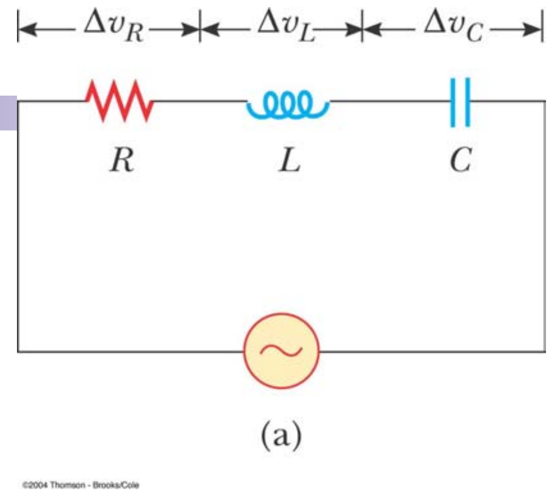
- The instantaneous voltage across each of the three circuit elements can be expressed as

$$\Delta v_R = I_{\max} R \sin \omega t = \Delta V_R \sin \omega t$$

$$\Delta v_L = I_{\max} X_L \sin\left(\omega t + \frac{\pi}{2}\right) = \Delta V_L \cos \omega t$$

$$\Delta v_C = I_{\max} X_C \sin\left(\omega t - \frac{\pi}{2}\right) = -\Delta V_C \cos \omega t$$

RLC series circuit



$$\Delta v_R = I_{max} R \sin \omega t = \Delta V_R \sin \omega t$$

$$\Delta v_L = I_{max} X_L \sin\left(\omega t + \frac{\pi}{2}\right) = \Delta V_L \cos \omega t$$

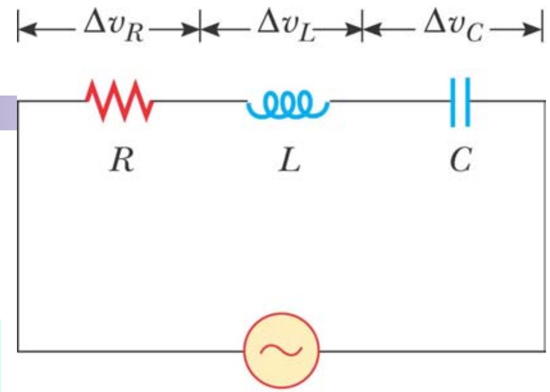
$$\Delta v_C = I_{max} X_C \sin\left(\omega t - \frac{\pi}{2}\right) = -\Delta V_C \cos \omega t$$

- In series, **voltages add** and the instantaneous voltage across all three elements would be

$$\Delta v = \Delta v_R + \Delta v_L + \Delta v_C$$

  – Easier to use the phasor diagrams

RLC series circuit



(a)

$$i = I_{\max} \sin \omega t$$

$$\Delta v_R = I_{\max} R \sin \omega t = \Delta V_R \sin \omega t$$

$$\Delta v_L = I_{\max} X_L \sin\left(\omega t + \frac{\pi}{2}\right) = \Delta V_L \cos \omega t$$

$$\Delta v_C = I_{\max} X_C \sin\left(\omega t - \frac{\pi}{2}\right) = -\Delta V_C \cos \omega t$$

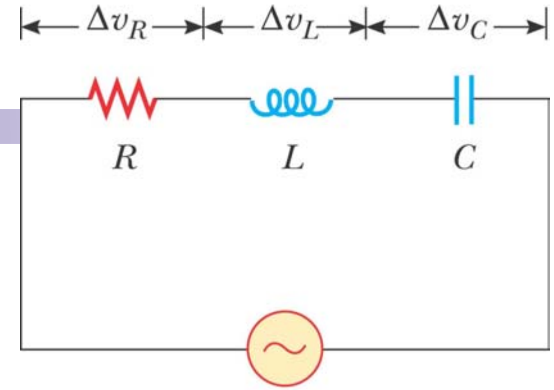
$$\Delta v = \Delta v_R + \Delta v_L + \Delta v_C =$$

$$= \Delta V_R \sin \omega t + \Delta V_L \cos \omega t - \Delta V_C \cos \omega t =$$

$$= \Delta V_{max} \sin (\omega t + \varphi)$$

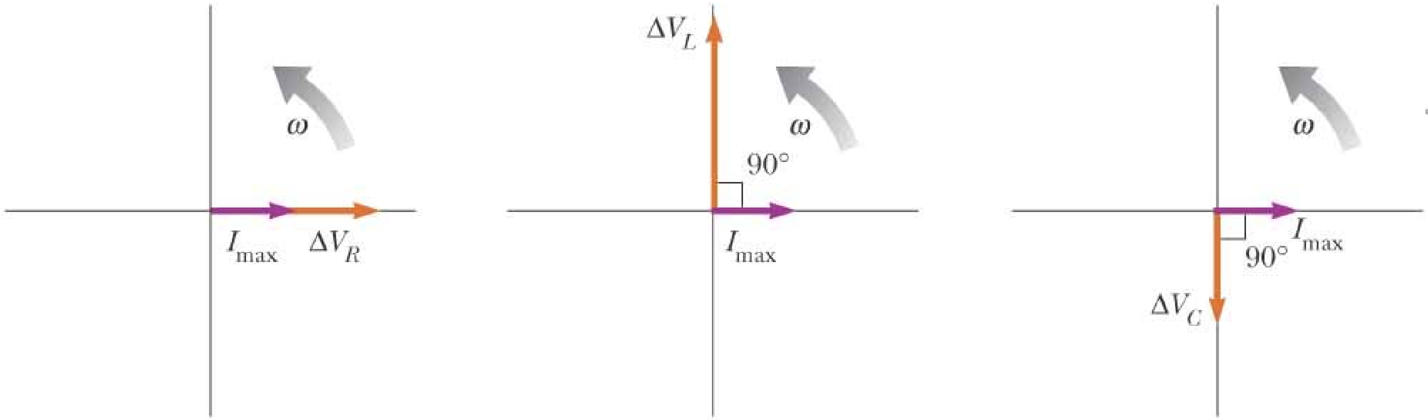
Easier to use the phasor diagrams

RLC series circuit



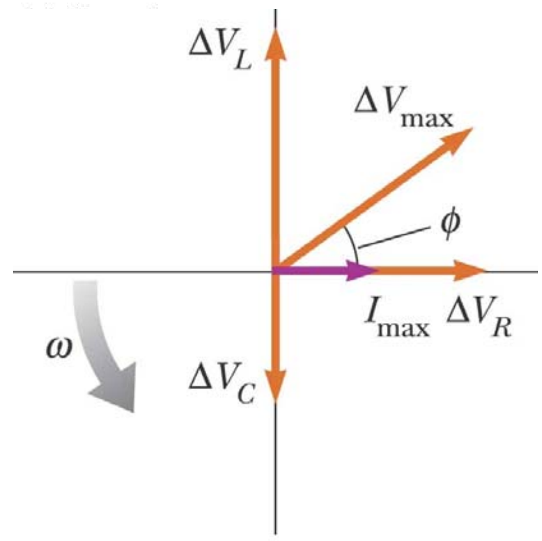
(a)

The phasors for the individual elements:



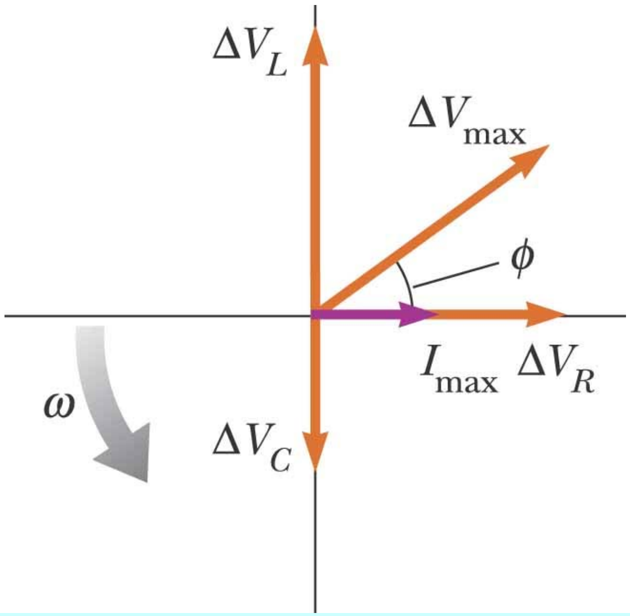
(a) Resistor (b) Inductor

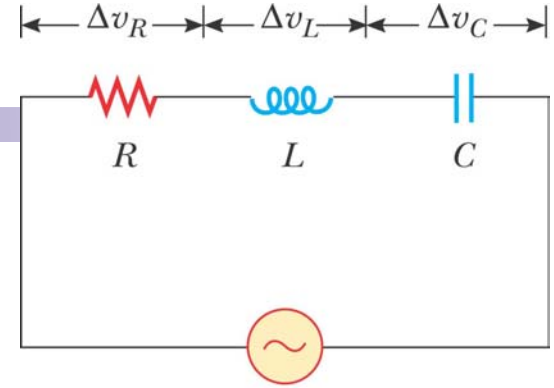
- The individual phasor diagrams can be combined
- Here a single phasor I_{max} is used to represent the current in each element
  - In series, the current is the same in each element



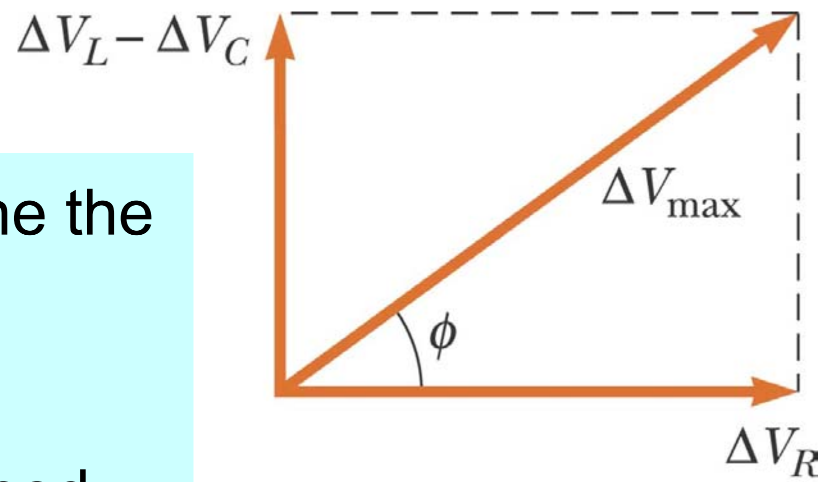
(a)

RLC series circuit





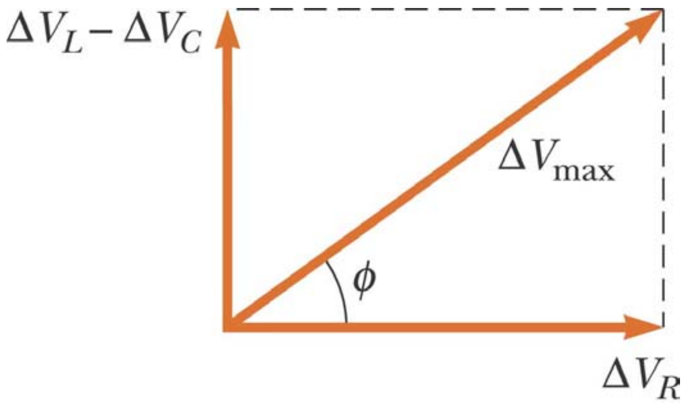
(a)



(b)

- Vector addition is used to combine the voltage phasors
- ΔV_L and ΔV_C are in opposite directions, so they can be combined
- Their resultant is perpendicular to ΔV_R

RLC series circuit



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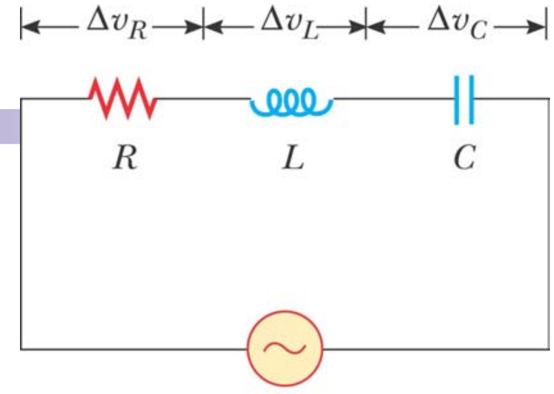


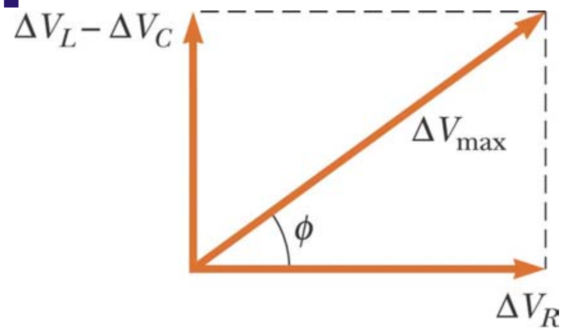
(a)

- From the vector diagram, ΔV_{max} can be calculated

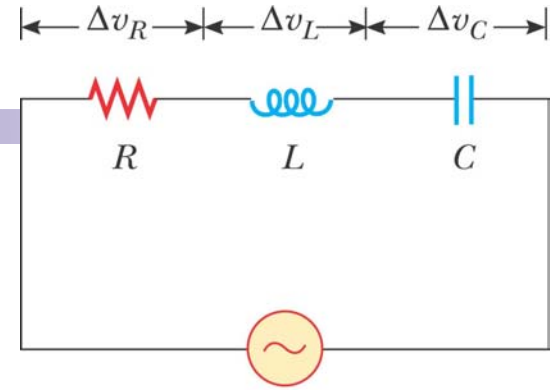
$$\Delta V_{max} = \sqrt{\Delta V_R^2 + \left(\Delta V_L - \Delta V_C\right)^2} = \sqrt{(I_{max} R)^2 + \left(I_{max} X_L - I_{max} X_C\right)^2}$$

$$\Delta V_{max} = I_{max} \sqrt{R^2 + \left(X_L - X_C\right)^2}$$

RLC series circuit



(b)



(a)

$$\Delta V_{max} = I_{max}\sqrt{R^2 + (X_L - X_C)^2}$$

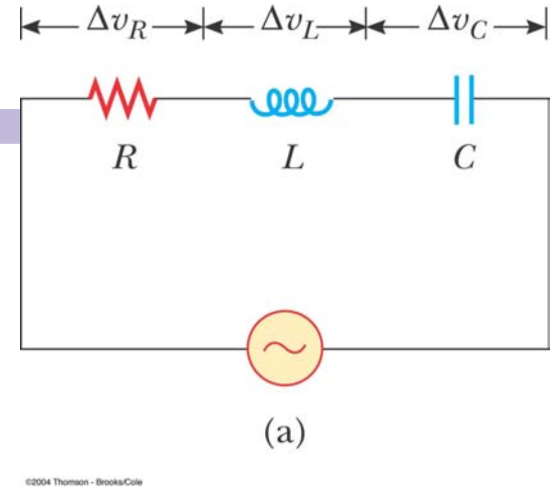
- The current in an RLC circuit is

$$I_{max} = \frac{\Delta V_{max}}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{\Delta V_{max}}{Z}$$

- ***Z*** is called the impedance of the circuit and it plays the role of resistance in the circuit, where

$$Z \equiv \sqrt{R^2 + (X_L - X_C)^2}$$

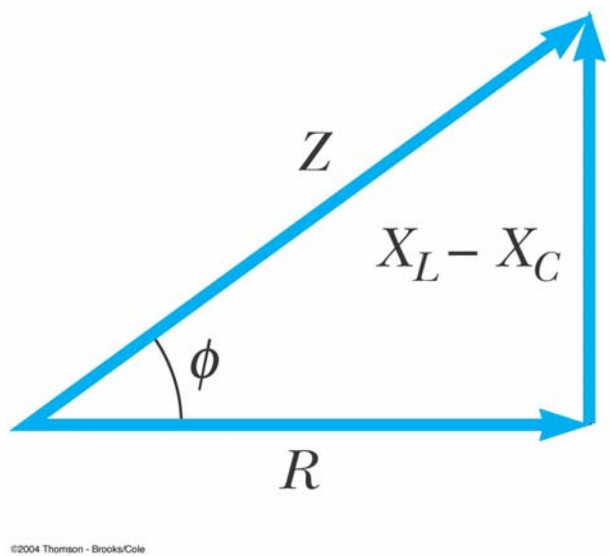
RLC series circuit



$$I_{\max} = \frac{\Delta V_{\max}}{Z}$$

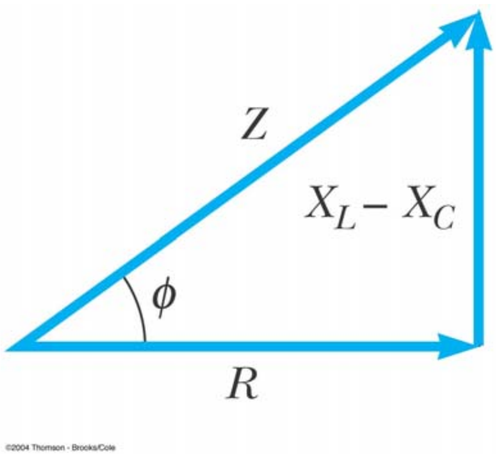
$$Z \equiv \sqrt{R^2 + (X_L - X_C)^2}$$

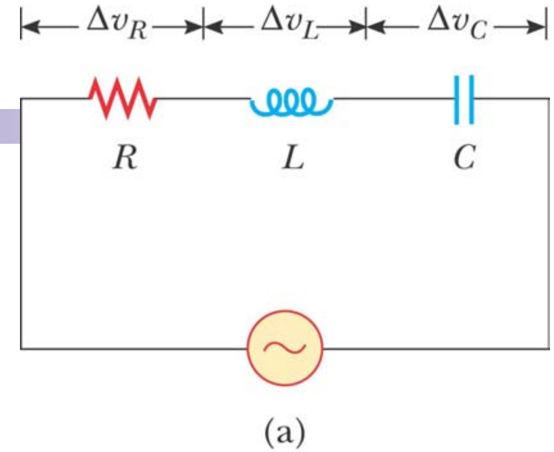
impedance triangle



RLC series circuit: impedance triangle

$$Z \equiv \sqrt{R^2 + (X_L - X_C)^2}$$





- The impedance triangle can also be used to find the phase angle, φ

$$\varphi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

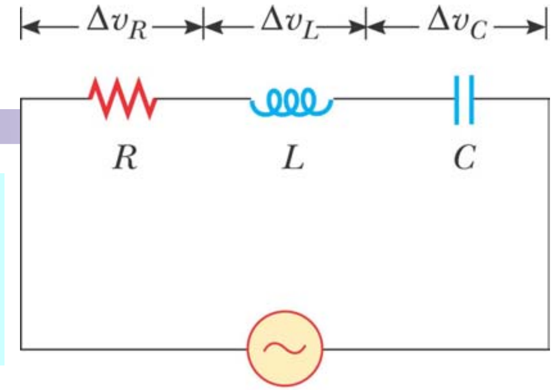
- The phase angle can be positive or negative and determines the nature of the circuit
- Also, $\cos\varphi = \dfrac{R}{Z}$

$$i = I_{max} \sin \omega t \qquad \Delta v = \Delta V_{max} \sin(\omega t + \varphi)$$

RLC series circuit

$$Z \equiv \sqrt{R^2 + (X_L - X_C)^2}$$

$$\varphi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$



(a)

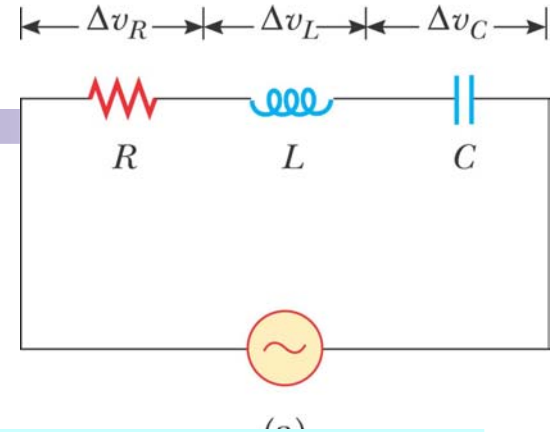
Table 33.1

Impedance Values and Phase Angles for Various Circuit-Element Combinations[a]

Circuit Elements	Impedance Z	Phase Angle ϕ
R	R	0°
C	X_C	−90°
L	X_L	+90°
R C	$\sqrt{R^2 + X_C^2}$	Negative, between −90° and 0°
R L	$\sqrt{R^2 + X_L^2}$	Positive, between 0° and 90°
R L C	$\sqrt{R^2 + (X_L - X_C)^2}$	Negative if $X_C > X_L$ Positive if $X_C < X_L$

[a] In each case, an AC voltage (not shown) is applied across the elements.

Power in AC circuit



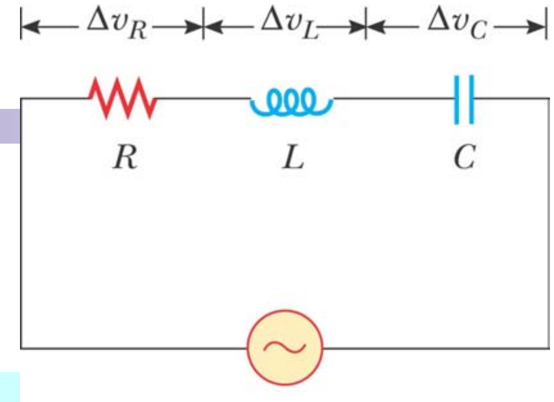
$$I_{\max} = \frac{\Delta V_{\max}}{Z} \qquad I_{rms} = \frac{I_{\max}}{\sqrt{2}}$$

- The average power delivered by the generator is converted to internal energy in the resistor
  - $P_{av} = \frac{1}{2} I_{max} \Delta V_{max} \cos \varphi = I_{rms} \Delta V_{rms} \cos \varphi$
  - $\cos \varphi$ is called the *power factor* of the circuit
- We can also find the average power in terms of *R*

$$P_{av} = I_{rms}^2 R = \frac{1}{2} I_{max}^2 R = \frac{1}{2}\left(\frac{\Delta V_{\max}}{Z}\right)^2 R = \frac{\Delta V_{max}^2}{2} \frac{R}{R^2 + (X_L - X_C)^2}$$

Resonances in AC circuit

$$P_{av} = \frac{\Delta V_{max}^2}{2}\frac{R}{Z^2} = \frac{\Delta V_{max}^2}{2}\frac{R}{R^2 + (X_L - X_C)^2}$$



(a)

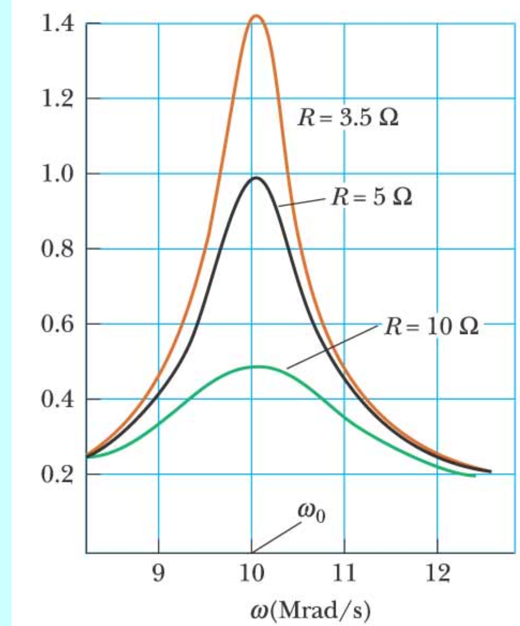
- *Resonance* in $P_{av}(\omega)$ occurs at the frequency ω_o where the current has its maximum value
- To achieve maximum current, the impedance must have a minimum value
  - This occurs when $X_L = X_C$ *or*

$$X_L = \omega_0 L = X_C = \frac{1}{\omega_0 C}$$

  - Solving for the frequency gives

$$\omega_o = 1/\sqrt{LC}$$

- The **resonance frequency** also corresponds to the natural frequency of oscillation of an *LC* circuit



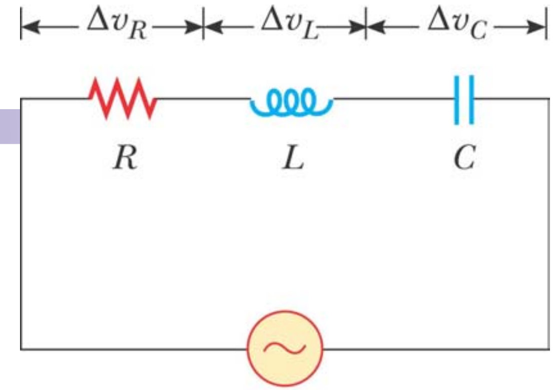
(a)

Resonances in AC circuit

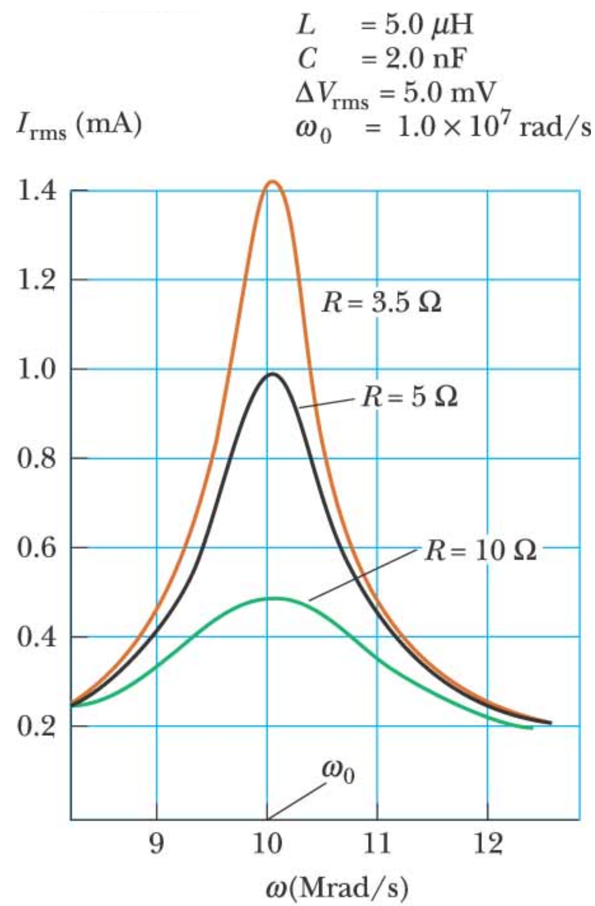
$$P_{av} = \frac{\Delta V_{max}^2}{2}\frac{R}{Z^2} = \frac{\Delta V_{max}^2}{2}\frac{R}{R^2 + (X_L - X_C)^2} = \frac{\Delta V_{max}^2}{2}\frac{R}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

$$\omega_o = 1/\sqrt{LC} \qquad P_{av}(\omega_0) = \frac{\Delta V_{max}^2}{2R}$$

- Resonance occurs at the same frequency regardless of the value of ***R***
- As ***R*** decreases, the curve becomes narrower and taller
- Theoretically, if ***R* = 0** the current would be infinite at resonance
  - Real circuits always have some resistance



(a)



(a)